Year 9 notes part 2:

- numbers 1.1 (p.1)
- fractions 2.1 (p.56)
- rounding, errors and estimates 3.3 (p. 123)
- percentages 3.5 (p.138)
- ratio & proportion 1.1 (p.6)

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Multiplying and dividing by numbers between 0 and 1.

Brief revision

The "something big" rules:

- If a number is multiplied by "something bigger than 1", it gets bigger (and vice-versa)
    - e.g. 5×1000 = 5000 (more than 5)
    - but 5 × 0.001 = 0.005 (less than 5)
- If a number is divided by "something bigger than 1", it gets smaller (and vice-versa)
    - e.g. 5÷1000 = 0.005 (less than 5)
    - but 5 ÷ 0.001 = 5000 (more than 5)

I know this was **obvious** to you back in year 5 – so why do people write answers that are *obviously wrong* and expect me to believe them? Check it is sensible!

When to multiply and divide (BIDMAS)

BIDMAS helps us remember the rules for evaluating a list of calculations. We need to know whether a sum written as 1+2×3 is meant to give the answer 9 (adding the 1 and 2 first) or 7 (multiplying the 2 and 3 first).

B	brackets
I	indices (squares, cubes, square roots etc)
DM	division and multiplication, working from left to right
AS	addition and subtraction, working from left to right.

Examples

1+2×3	do the multiplication first (2×3=6), then the addition (1+6=7): so NOT 9
$1+3^2$	do the squaring first (3^2=9), then the addition (9+1 = 10)
2×8÷4	left to right, 2×8=16, 16÷4=4
1+4-3+5	left to right, 1+4=5, 5-3=2, 2+5=7 or rearrange as 1+4+5-3 = 10-3 = 7
(1+2)×3	do the brackets first (1+2=3), then multiplication (3×3=9)

$12 \div 3 \times 2 = 4 \times 2 = 8$ "D before M"

$12 \div (3 \times 2) = 12 \div 6 = 2$ "B before D"

Addition

Write it down! I know you can do sums in your head but you are making life hard for yourself. It is quicker to use pen and paper and it means:

- you can go back and check it at the end of the exam
- the examiner can give you marks for working even if the final answer comes out wrong

Write the numbers so the decimal points line up (or the units column if no point). Add the right hand column, then the one to its left etc.

```
1234
 325 +
1559
```

How to carry:

When a column adds to more than 9, write its units digit and carry the tens digit as a little number in the next column to be added later, eg

```
 4785
 6517 +
   12
```

(think 5+7=12). Now add the next column, 8+1+1=10 so write 0 and carry 1.

```
 4785
 6517 +
  102
```

The next column has 7+5+1=13, write 3, carry 1.

```
 4785
 6517 +
 1302
```

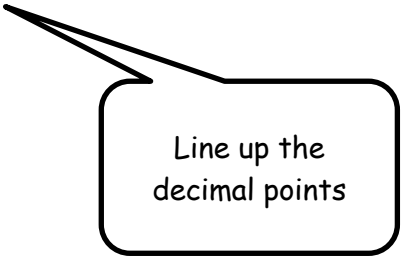
Finally 4+6+1=11, write 1, carry 1 but as there are no numbers left to add this becomes a proper 1 in the answer:

```
 4785
 6517 +
11302
```

Check you can add 3 or more numbers and/or decimals in the same way, eg

```
 132                    1.23
 405                    4.107 +
 789 +                  5.337
1326
```

Line up the decimal points



Subtraction

For a subtraction like 96-31, write as

```
96
31 -
65
```

"Borrowing" when subtracting a large digit from a small one

Sometimes we need to add 10 to a digit to help us subtract. We do this by borrowing a "1" from the next column to the left. *So we don't forget we have done this*, we cross out the original number and write the new version above.

Example: 742-259

Cannot take 9 from 2, so borrow 1 from the 4 (it becomes 3) and take 9 from 12:

```
 3
7 4 1 2
259 -
  3
```

Now cannot take 5 from 3 so borrow 1, take 5 from 13

```
 6  3
7 1 4 1 2
2 5 9 -
  83
```

Finally take 2 from 6:

```
 6  3
7 1 4 1 2
2 5 9 -
4 8 3
```

We can check our answer: if we add 259+483 we should get back to 742.

Borrowing from further left

Sometimes we have to borrow from beyond the next column.

To find 6005 – 787 I have to borrow 10 from the 6000. The 10 makes the 5 up to 15, so I can subtract 7, leaving 5990 behind in the other columns:

$$\begin{array}{r} {}^{5}\not{6}\,{}^{9}\not{0}\,{}^{9}\not{0}\,{}^{1}5 \\ 787 \quad - \\ \hline 5218 \end{array}$$

Subtracting a large number from a small one

If we subtract a large number from a small one, the answer will be a negative number. The easiest way is to subtract them the other way around, so the answer is positive, then make your final answer negative eg:

To find 259-742, show working for 742-259 = 483 as above, then write "hence 259-742 = -483".

Adding lists of positive and negative numbers

Add up all the positive numbers, add the negative numbers, then combine the answers eg:

"Find the sum of these numbers:

3 4 −12 18 −33 20 −5 "

We think 3+4+18+20 = 7+18+20 = 25+20 = 45 and -12 -33 -5 = -45-5 = -50

Then 45-50 = -5.

Multiplication

Again, we write the two numbers so the units columns line up vertically. It doesn't really matter which is the top number but it is easiest to put the number with the most non-zero digits on the top line.

To multiply 23×3, think 3×3=9,

```
23
 3 ×
 9
```

and then 2×3=6, moved across one column to the left:

```
23
 3 ×
69
```

Carrying

As with addition, if when we multiply digits the answer is >9 we must carry the tens digit into the next column. We write it small so we don't forget it.

Example: 57×7, first multiply the right-hand digit in the top number, 7×7=49

```
57
 7 ×
 4
49
```

Now multiply the next digit, 5×7=35, 35+4=39

```
 57
  7 ×
399
 4
```

Multiplying longer numbers

Put the number with the most non-zero digits on the top line. Each digit in the second number will generate its own line as part of the solution.

Example: "Multiply 23×16"

We can (as above) easily do 23×6=138 and 23×10=230 so we write two intermediate answers, then add them to get the final answer

```
 23
 16 ×
138
230 +
368
```

Note how when we multiply 23×10, the 23 shifts <u>to the left by 1 column</u> because the 1 is in the tens column, one place to the left of the units column.

Example

"Multiply 234×321"

This time we will have 3 intermediate answers (234×1, 234×20, 234×300) to add up

```
  234
  321 ×
  234
 4680
70200 +
75114
```

<u>Multiplying by powers of 10</u>

To multiply by any power of 10, move the decimal point to the right or left.

Examples

`12.3×1000 = 12300.` (multiplying by 10^3, the decimal point moves 3 places to the right)

`9.71×0.01 = 0.0971` (multiplying by 10^{-2}, the decimal point moves 2 places to the left)

<u>Multiplying decimal numbers</u>

Just the same, but your answers move to the right when multiplying by digits to the right of the units column (or use the lattice method, below).

Example

"Multiply 2.4×0.12"

```
2.4
0.12 ×
0.24      2.4 moves 1 place to right when multiplied by 0.1
0.048 +   4×2=8, move 2 places to right as the 2 is two to right of the units column
0.288
```

The lattice method

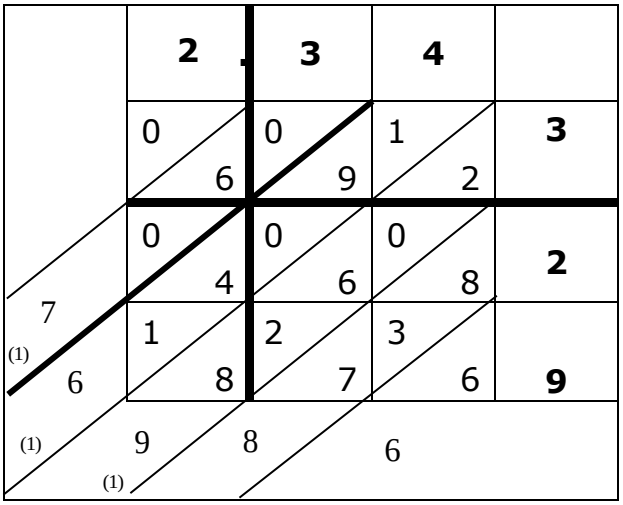
- Write the digits along the top and right sides of a grid, splitting each cell in the grid diagonally. *Decimal points go on a grid line*.
- Multiply each pair of digits, splitting the answer between left and right sides of the cell (if the answer <10, put a 0 in the left side).
- Now add up along each diagonal
- Put a decimal point where the two decimal point grid lines intersect, then follow this down along a diagonal line to find the decimal point position in the answer.
- Add up the digits between the diagonals, starting at bottom right and carrying as appropriate.
- Read off the answer.

Example

"Multiply 2.34×3.29" Note the lines showing the decimal point position. The answer is 7.6986. We finish by checking:

(i) the number is about the right size – I estimate a bit bigger than 2×3=6, so am happy with 8 (not 80, not 0.8).

(ii) having started with 4 digits to the right of the decimal points (.34 and .29), I must finish with 4 digits to the right of the decimal point (.6986).



The decimal point rule lets us do simple multiplications as mental arithmetic:

0.3×0.04 = 0.012

- Must have the digits 12 and also 3 digits to right of the decimal point.
- Also check it is <u>less than</u> 0.04
  (since 0.04 was multiplied by a number <u>less than 1</u>)

Division

<u>Short division</u>

Example "Find 7854÷3" (how many 3's go into 7854)

First we swap the order, so the 3 is on the left:

$$3\overline{)7\ 854}$$

Now we divide 7÷3 to get 2 and write this above the 7 as part of the answer.

2×3=6 not 7, there is 1 left so we write this in front of the 8 to make an 18:

$$3\overline{)\begin{array}{r}2\\ 7^{1}854\end{array}}$$

Now we do 18÷3 = 6, write this as the next part of the answer: (No remainder to carry).

$$3\overline{)\begin{array}{r}2\ 6\\ 7^{1}854\end{array}}$$

5÷3 = 1 remainder 2, write the 1 above, put a small 2 in front of the 4 to make 24:

$$3\overline{)\begin{array}{r}2\ 61\\ 7^{1}85^{2}4\end{array}}$$

Finally, 24÷3 = 8:

$$3\overline{)\begin{array}{r}2\ 61\ 8\\ 7^{1}85^{2}4\end{array}}$$

Answer 2618.

Example

Find 59.5 ÷ 0.7 (how many 0.7's go into 59.5)

We would prefer to divide by 7, not 0.7, so first "balance" the calculation by multiplying each side by 10, to make it 595 ÷ 7

5 does not divide by 7 but we know 7×8=56 so 56÷7 = 8. Then 59÷7 = 8 remainder 3:

$$7\overline{)59^{3}5}\quad 8$$

Now we do 35÷7 = 5:

$$7\overline{)59^{3}5}\quad 8\ 5$$

Answer 85.

Long division

The principle is just the same as for short division but we write out the subtractions in rows to make it easier when using multi-digit numbers.

- See the long division PowerPoint.

Example "Find 157.5÷7"

First we swap the order, so the 7 is on the left:

$$7\overline{)157.5}\quad 22.5$$

$140 -$ Think $20\times 7=140$, put 2 in tens column of answer, write 140 to subtract

17.5

$14 -$ Think $2\times 7=14$, put 2 in units column of answer, write 14 to subtract

3.5

$3.5-$ Think $0.5\times 7=3.5$, put 5 in tenths column of answer, write 3.5 to subtract

0

Answer 22.5

Example "Find 5301÷31"

First we swap the order, so the 31 is on the left:

$$
\begin{array}{r}
171 \\
31\overline{)5301} \\
\underline{3100\ -} \\
2201 \\
\underline{2170\ -} \\
31 \\
\underline{31-} \\
0
\end{array}
$$

Think $100 \times 31 = 3100$, put 1 in hundreds column of answer, write 3100 to subtract

Think $70 \times 31 = 2170$, put 7 in tens column of answer, write 2170 to subtract

Think $1 \times 31 = 31$, put 1 in units column of answer, write 31 to subtract

Remainder 0 so we've finished.

Answer 171

Rules of fractions

Equivalence and simplification

A fraction does not change in value if the top ("numerator") and bottom ("denominator") are each multiplied by the same value:

$\frac{2}{3}=\frac{4}{6}=\frac{8}{12}$ (multiplying top and bottom by 2 each time)

Also true if top and bottom are divided by the same value: $\frac{3}{9}=\frac{1}{3}$

nb: Under no circumstances should you add anything to top and bottom of a fraction – this changes its value so whatever relationship you started with would no longer be true. Think $\frac{2}{3}\neq\frac{1}{2}\neq\frac{0}{1}$ ($\neq$ means "not equal to").

Converting mixed numbers to improper fractions and vice-versa.

To convert mixed to improper, write the integer part as a fraction (so both denominators are the same), then add:

$$3\frac{1}{4}=3\times\frac{4}{4}+\frac{1}{4}=\frac{12+1}{4}=\frac{13}{4}$$

3 is 3×1 and 1 is $\frac{4}{4}$

To convert an improper fraction to a mixed number, write the numerator (top) as a *multiple of the denominator* + a remainder, eg:

$$\frac{22}{3}=\frac{21+1}{3}=\frac{7\times3+1}{3}=7\frac{1}{3}$$

(think "*how many times does 3 go into 22? - 7 times, remainder 1*")

Addition

Same denominator – just add the numerators ("tops") and integer parts, if any.

$$\frac{1}{5}+\frac{2}{5}=\frac{1+2}{5}=\frac{3}{5}$$

$$2\frac{1}{5}+3\frac{2}{5}=5+\frac{1+2}{5}=5\frac{3}{5}$$

Addition – different denominators.

Find the lowest common multiple (LCM) of the two denominators and express each fraction in terms of it before adding and simplifying

$$\frac{4}{15}+\frac{7}{30}=\frac{8}{30}+\frac{7}{30}=\frac{15}{30}=\frac{1}{2}$$

The 30 here is the LCM of 15 and 30. We could equally well use 15×30 = 450 but all the numbers get very large so it is nicer to use the smallest possible common multiple.

Remember how to find a lowest common multiple? Multiply the two numbers and divide by their highest common factor: $\frac{15\times 30}{15}=30$

Subtraction

Subtraction uses a similar method: $\frac{3}{5}-\frac{2}{5}=\frac{3-2}{5}=\frac{1}{5}$

$\frac{2}{3}-\frac{1}{5}=\frac{10}{15}-\frac{3}{15}=\frac{10-3}{15}=\frac{7}{15}$ (15 is the LCM of 3 and 5)

Multiplication.

Very easy – "multiply the numerators (tops), multiply the denominators (bottoms)":

$$\frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}$$

but with mixed numbers, must convert to improper fractions first – see (b) above.

$$1\frac{1}{3}\times 3\frac{1}{5}=\frac{4}{3}\times\frac{16}{5}=\frac{64}{15}=4\frac{4}{15}$$

To multiply by an integer, just multiply the numerator by it: $2\times\frac{2}{3}=\frac{4}{3}$

You must know what fraction multiplication means:

$\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$ means "if I have half a cake, and cut it in 4 equal parts, one of these parts is $\frac{1}{8}$ of a cake".

Remember, multiplying by $\frac{1}{4}$ is the same as dividing by 4.

Reciprocals

Find the reciprocal of a fraction by swapping the numerator and denominator

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ (which means that $\frac{3}{2}=\frac{1}{\left(\frac{2}{3}\right)}$)

The reciprocal of 2 is $\frac{1}{2}$ and vice-versa.

The reciprocal of x is $\frac{1}{x}$ (use x^{-1} button on calculator).

Dividing fractions

To divide by a fraction, multiply by its reciprocal (so turn the second fraction upside down, then multiply)

$\frac{3}{4}\div\frac{1}{8}=\frac{3}{4}\times\frac{8}{1}=\frac{24}{4}=6$ (8 is the reciprocal of $\frac{1}{8}$)

Dividing fractions, easy as pie!
Flip the second and multiply.

You must understand what this means: "if I have $\frac{3}{4}$ cake and people want 1/8 cake each, I can feed 6 people" (the "filling containers" idea of division) or "How many eighths go into three quarters?"

This is similar to the calculation $12\div 2=6$ which means "if I have 12 cakes and people want 2 each, I can feed 6 people"

To divide by an integer, multiply the denominator by it (or multiply the fraction by the number's reciprocal as above) eg

$\frac{1}{2}\div 3=\frac{1}{2\times 3}=\frac{1}{6}$ or $\frac{1}{2}\div 3=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

Percentages

Ratios

Ratios enable us to divide collections of objects into groups or to describe the relative size of groups

- The ratio 1:2 shows that out of a total of 3 "parts", I have 1 of one kind and 2 of another.
- The ratio 1:2:4 shows that out of a total of 7 "parts", I have 1 of one kind and 2 of another and 4 of a third kind.

Example

30 people are split into 2 groups, A and B, in the ratio 10:20.

How many are in each group?

- A has 10, B has 20.

Simplest form

Just like equivalent fractions, we can multiply or divide all the numbers in a ratio by a constant without changing their relative proportions.

e.g. 2:3 = 4:6 = 6:9 etc

To reduce to the simplest form, divide each side by any common factor.

e.g. 24:20 has a common factor of 4 (it is 6×4 : 5×4), we divide each side by 4 to get 6:5

Finding components when the total is given

Example

20 people are split into 2 groups C and D in the ratio 9:1. How many are in each group?

- Add the ratio numbers, 9+1 = 10 "parts".
- 20 people ÷ 10 parts = 2 people per part.
- Multiply the ratio by the size of a part, [9:1] * 2 people = 18:2.
  - We have 18 people in group C and 2 in group D.

Example

A sponge cake recipe specifies flour, margarine and sugar in the ratio 4:2:1 by weight.

I want to make a 3.5kg cake. How much of each ingredient do I need?

- Add up the ratio numbers, 4+2+1 = 7 parts
- Divide total mass by number of parts to get the mass per part, 3.5 ÷ 7 = 0.5 kg/part.
- Multiply the ratio numbers by the mass per part, [4:2:1] ×0.5 gives
  - 4×0.5 = 2 kg flour
  - 2×0.5 = 1 kg margarine
  - 1×0.5 = 0.5 kg sugar

Finding the total when a component is given

Example

Stainless steel contains iron, chromium and nickel in the ratio 74:18:8

(a) If I have 90 kg chromium, how much stainless steel can I make?

- Add up the parts: 74+18+8 = 100
- Total mass = $\frac{100}{18}\times 90 = \frac{9000}{18} = 500$ kg

(or think: mass of 1 part = $\frac{90}{18} = 5$ kg, therefore 100 parts = 500 kg).

(b) How much nickel do I need?

Either:

Calculate total mass = 500 kg. $\frac{8}{100}$ of this is nickel, mass $\frac{8}{100}\times 500 = 8\times 5 = 40$ kg

Or

Each part = 5 kg, 8 parts = 40 kg.

Direct proportion

Two quantities are said to be in *direct proportion* when the ratio of one to the other is constant. We can also say that one is <u>proportional to</u> the other.

- A recipe for syrup pudding includes 100g flour and 50g sugar. Regardless of how many puddings we make (multiples of the recipe), we know the amount of flour ***f*** is proportional to the amount of sugar ***s***. If one doubles, so must the other.

Similarly, if I enlarge a photograph, the length of the short side will be proportional to the length of the long side.

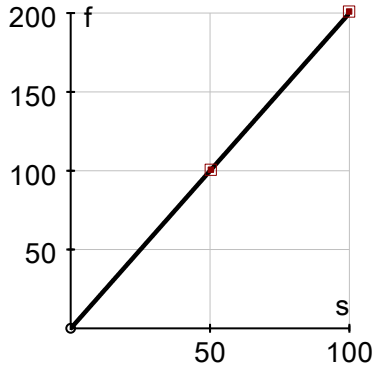
Double the width, double the height

If the short side is not proportional to the long side, the photo gets distorted:

(photo from Wikipedia)

Instead of an = sign, we have a special sign to indicate "is proportional to": $f \propto s$

This implies that the graph of ***f*** against ***s*** would be a straight-line through the origin with some gradient ***k*** (equation $f = ks$):



Typically a question will give you a pair of coordinates on the line. This tells us the gradient of the line.

e.g. A recipe includes 100g flour and 50g sugar.
How much flour will I need if I am going to use 100g sugar?

There are 3 ways of thinking about this – they are all good, but one approach is particularly good as it lets us do more complicated problems later.

(1) Getting the formula by finding the gradient.

$\text{gradient } k = \dfrac{\text{up}}{\text{across}} = \dfrac{100}{50} = 2$, hence the formula is $f = 2s$.

Then put $s = 100$ into the formula to get $f = 2 \times 100\text{g} = 200\text{g}$

nb (*nota bene*): be careful with the units of your answer!

(2) the "unitary" method: how much flour do we need for 1 gram of sugar?

For 1 g sugar I need $\dfrac{100\text{g}}{50} = 2\text{g}$ flour, so then for 100 g sugar I need $2\text{g} \times 100 = 200\text{g}$ flour.

Hey! That's the same as the gradient method, I just didn't call it gradient!

(3) The "scaling up" or "enlargement factor" method. (**best way**!)

What multiple of the recipe am I making?

I am making $\dfrac{100}{50} = 2\times$ the recipe so I need $2 \times 100\text{g} = 200\text{g flour}$.

$\left(\dfrac{100}{50}\right)$ is the "enlargement

<u>Examples</u>

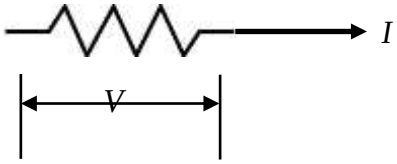
(a) the power a solar cell can produce is proportional to its surface area.

A cell with 0.3 m^2 area produces 15 Watts.

- What area will I need to generate 300 Watts?
  - If I want to generate 300 Watts, I will need $\dfrac{300\text{ W}}{15\text{ W}} \times 0.3\text{ m}^2 = 6\text{ m}^2$
- If I have 24 m^2, what power will it generate?
  - $\dfrac{24\text{ m}^2}{0.3\text{ m}^2} \times 15\text{ W} = 1200\text{ W}$

Note: The really nice thing about this method is that the enlargement factors $\dfrac{300\text{ W}}{15\text{ W}}$ or $\dfrac{24\text{ m}^2}{0.3\text{ m}^2}$ are <u>dimensionless</u> (just a number, no units). The Watts or m^2 cancel out top and bottom. We then automatically keep the correct units when we scale up the 0.3 m^2 or the 15 W.

(b) The current I through a resistor is proportional to the voltage V across it.



When V = 20 volts, I = 4 amps.

Because we know about proportion, you do not need to have learnt about voltage and current – it is "just a proportion question".

What is the current I if V = 240 Volts?

$$I = \frac{240 \text{ volts}}{20 \text{ volts}} \times 4 \text{ amps} = 48 \text{ amps}$$ ("12× the voltage so 12× the current")

What is the voltage if the current is 1 amp?

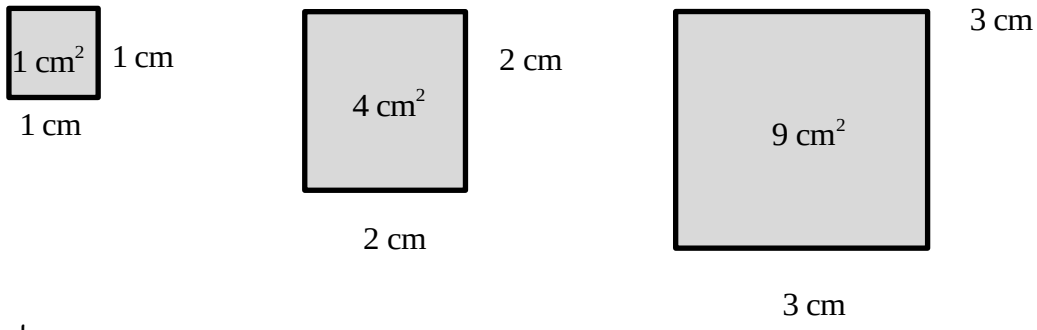
$$V = \frac{1 \text{ amp}}{4 \text{ amps}} \times 20 \text{ volts} = 5 \text{ volts}$$ ("¼ the current so ¼ the voltage")

<u>Area and volume scaling</u>

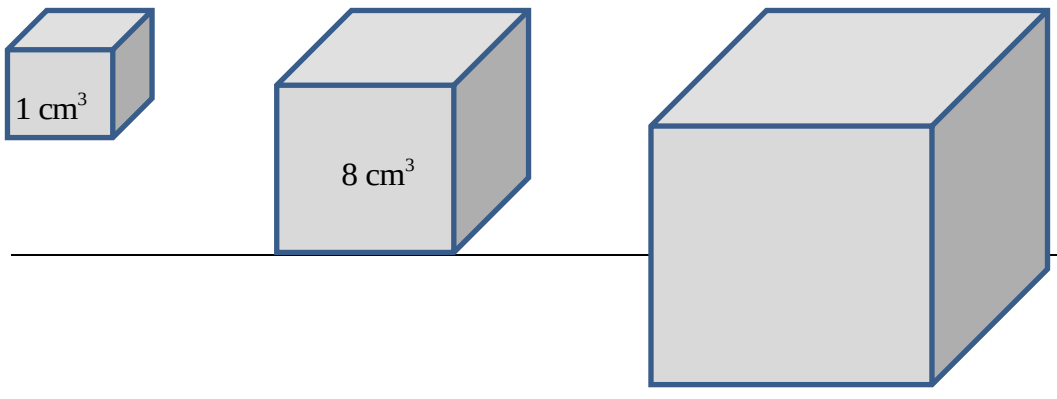
When an object is enlarged to make a new object that is "geometrically similar" (all angles same as before, all lengths in the same proportion):

- area $\propto$ (enlargement factor)^2
- volume $\propto$ (enlargement factor)^3

e.g. squares:



cubes:



$27\ \text{cm}^3$

Remember:

- $100\ \text{cm} \times 100\ \text{cm} = 10000\ \text{cm}^2 = 1\ \text{m}^2$
- $1000\ \text{m} \times 1000\ \text{m} = 1000000\ \text{m}^2 = 1\ \text{km}^2$
- $10\ \text{mm} \times 10\ \text{mm} \times 10\ \text{mm} = 1000\ \text{mm}^3 = 1\ \text{cm}^3$

Examples:

(i) A drawing of a garden shows a pond, area 12 cm^2 on the paper. The scale of the drawing is 1:50. What is the area of the pond?

Area = $50^2 \times 12\ \text{cm}^2 = 30000\ \text{cm}^2 = 3\ \text{m}^2$.

(ii) A ship 20 m long has a volume of 150 m^3. A bigger version of the ship is 30 m long. What is its volume?

volume = $\left(\frac{30\ \text{m}}{20\ \text{m}}\right)^3 \times 150\ \text{m}^3 = 1.5^3 \times 150\ \text{m}^3 = 506.25\ \text{m}^3$