Unit 3 Proportion

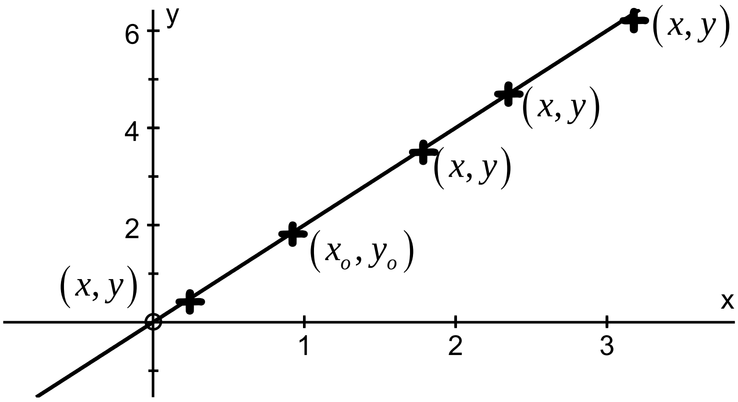
Direct proportion

$\propto$ means "is proportional to" (another sign like =, <, > etc).

$y \propto x$ means that if x doubles, then y doubles.

We could write a formula $y = kx$.

This defines a straight line through the origin and through some "original" point $(x_o y_o)$. All other pairs of (x,y) points lie on this line.



It is more useful to think of an "enlargement factor". If x doubles, the enlargement factor = 2. This makes y double. So we can write:

$$y = (\text{enlargement factor}) \times \text{original y}$$

therefore: $y = \left(\frac{x}{x_o}\right) \times y_o$

- The enlargement factor may be >1 or <1. It can even be negative!

Example.

y = 20 when x = 4 and $y \propto x$.

(a) What is y when x = 40?, (b) what is the formula y = kx?

(a) $y = \left(\frac{40}{4}\right) \times 20 = 10 \times 20 = 200$

(b) To get a formula, just put a general value "x" instead of the 40:

$$y = \left(\frac{x}{4}\right) \times 20 = \left(\frac{20}{4}\right) x = 5x$$

Other kinds of proportion.

Proportion occurs because of a physical formula

- $F = ma$ hence $F \propto a$
- $KE = \frac{1}{2}mv^2$ hence $KE \propto v^2$
- Gravitational force $F = \frac{Gm_1m_2}{r^2}$ hence $F \propto \frac{1}{r^2}$

We can do any kind of proportion question by simply raising the enlargement factor to some power.

Example

10 acre fields have a length that is inversely proportional to their width, $L \propto \frac{1}{w}$, which we can write as $L \propto w^{-1}$.

A field of length 440 yards will have width 100 yards.

How long would a field be, if the width was 55 yards?

$$L = \left(\frac{55}{100}\right)^{-1} \times 440 \text{ yards}$$

Enlargement factor to power -1

Calculator!

If you want a formula, $L = \left(\frac{w}{100}\right)^{-1} \times 440 \text{ yards}$

$$= \left(\frac{100}{w}\right) \times 440 \text{ yards} = \frac{44000}{w} \text{ yards}$$

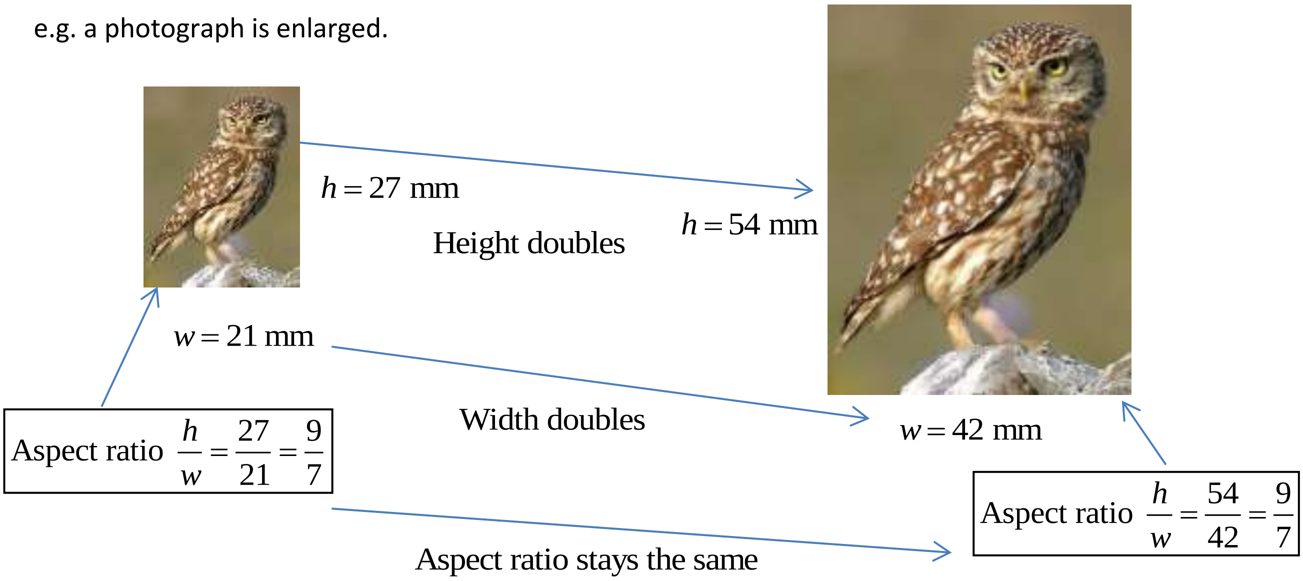
Geometric similarity

Some questions expect you to identify a suitable proportion formula. These usually involve **geometric similarity**.

Two shapes (two- or three dimensional, triangles, rectangles, cuboids, prisms, pyramids etc) are geometrically similar if one is an enlargement of the other.

- They look the same (apart from size)
- The ratio of any pair of sides in one shape = the equivalent ratio in the other shape
- The enlargement factor (side of shape A) to (side of shape B) is constant, whichever equivalent pair of sides we choose
- Each angle in shape A is the same as the equivalent angle in shape B.

e.g. a photograph is enlarged.



Whenever we have an enlargement (i.e. "geometric similarity") we can say:

- ➢ $\text{area} \propto \text{length}^2$ (and $\text{length} \propto \sqrt{\text{area}}$)
- ➢ $\text{volume} \propto \text{length}^3$ (and $\text{length} \propto \sqrt[3]{\text{volume}}$)
- ➢ $\text{volume} \propto \text{area}^{\frac{3}{2}}$ (and $\text{area} \propto \text{volume}^{\frac{2}{3}}$)

The proportion formula must use a corresponding pair of lengths (e.g. 21, 42) and a corresponding pair of areas or volumes. It works for any kind of area (area of a shape, cross-sectional area, surface area).

Example

A boat of length 5 m has sails of area 12 m^2. $\text{Area} \propto \text{length}^2$.

A geometrically similar boat is 8 m long. What sail area does it have?

$$A = \left(\frac{8}{5}\right)^2 \times 12 \text{ m}^2 = 30.72 \text{ m}^2$$

Backwards proportion

If we know $A \propto L^2$ ($\text{area} \propto \text{length}^2$) what is the proportion relation for length in terms of area?

- I can square root each side: $\sqrt{A} \propto L$
- I can swap sides: $L \propto \sqrt{A}$

e.g. An egg with a surface area of 30 cm^2 is 40 mm long.

How long is a geometrically similar egg with surface area 40 cm^2?

Knowing $L \propto \sqrt{A}$ for any pair of geometrically similar objects, I can write:

$L = \sqrt{\left(\frac{40}{30}\right)} \times 40 \text{ mm}$. Putting this in my calculator gives *L* = 46.2 mm.

Finding a constant "k"

Questions often tell us that a proportion relationship obeys a formula such as $y = kx$ or $L = k\sqrt{A}$ and ask us to find the "constant of proportionality" *k*.

You can do these in two ways:

- Just substitute a pair of values and solve an equation for *k*, or
- Write a proportion formula using a letter in place of the new *x* value.

Examples:

(a) $y = kx$ and *y* = 20 when *x* = 4. <u>Find *k*</u>.

- Substitute *y* = 20, *x* = 4; solve $20 = k \times 4$ to get $k = \frac{20}{4} = 5$
- Since $y \propto x$ I could write: $y = \left(\frac{x}{4}\right) \times 20$.

  This simplifies to $y = 5x$, which is $y = kx$ with $k = 5$.

(b) $L = k\sqrt{A}$ and *L* = 40 mm when *A* = 30 cm^2. <u>Find *k*</u>.

It is very sensible to use common units (cm, cm^2) to avoid confusion later (though this is not always possible, consider $F = \frac{k}{r^2}$ for gravitational attraction)

- Substitute L = 4 cm, A = 30 cm^2; solve $4 = k\sqrt{30}$ to get $k = \frac{4}{\sqrt{30}} = 0.7303$
- Alternatively write $L = \sqrt{\left(\frac{A}{30}\right)} \times 4 \text{ cm} = \frac{4}{\sqrt{30}}\sqrt{A}$, hence $L = k\sqrt{A}$ with

  $k = \frac{4}{\sqrt{30}} = 0.7303$

On the whole, unless the question asks you to find k it is much nicer to use the enlargement factor method. The numbers are exact (did you round k? Oh no!) and you automatically get the correct units.

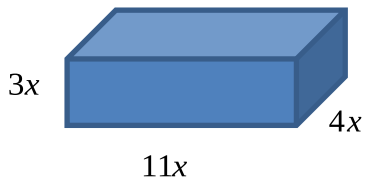
"Size of a brick" questions.

This is a kind of proportion question where we must choose length, width and height in some proportion. You can do it in two ways:

a) Writing an equation and solving it
b) Finding the volume of a basic brick, then enlarging it

e.g. The famous architect Arne Jacobsen specified when he designed St Catherine's college that the bricks should be made longer and thinner than usual. He wanted a length: height:width in the ratio 11:3:4. If each brick has a volume of 1000 cm^3, what are the dimensions?

(a)



I will define each length as a multiple of an unknown length "x". The 11x, 3x, 4x multiples make them in the correct proportion. Now we write an equation:

$$11x \times 3x \times 4x = 1000 \text{ cm}^3$$

$$132x^3 = 1000 \text{ cm}^3$$

$$x = \sqrt[3]{\frac{1000}{132}} = 1.964 \text{ cm}$$

The sides are then 21.6 cm, 5.9 cm, 7.9 cm.

(b) Alternatively I can imagine a basic brick with sides 11 cm, 3 cm, 4 cm, find its volume, then scale it up.

$$11 \times 3 \times 4 = 132 \text{ cm}^3$$

Knowing $\text{length} \propto \sqrt[3]{\text{volume}}$, I need to enlarge each side by factor of $\sqrt[3]{\frac{1000}{132}}$.

The sides are then $11\times\sqrt[3]{\frac{1000}{132}} = 21.6\text{ cm}$, $3\times\sqrt[3]{\frac{1000}{132}} = 5.9\text{ cm}$, $4\times\sqrt[3]{\frac{1000}{132}} = 7.9\text{ cm}$.