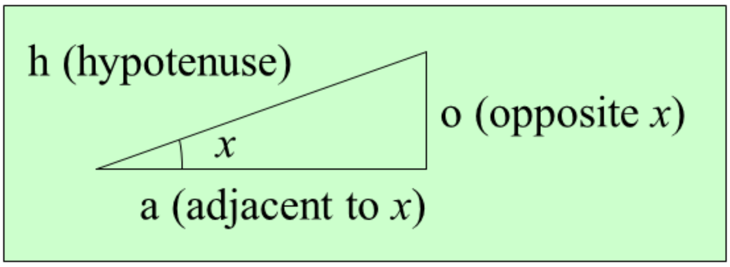
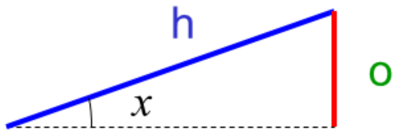
Trigonometry

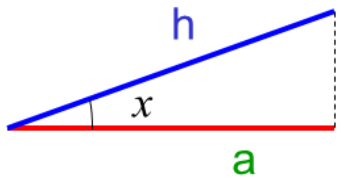
Soh-Cah-Toa





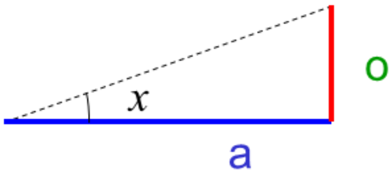
$$\sin x = \frac{o}{h} \quad \text{(soh)}$$

$$o = h \sin x$$
$$h = \frac{o}{\sin x}$$
$$x = \sin^{-1}\left(\frac{o}{h}\right)$$



$$\cos x = \frac{a}{h} \quad \text{(cah)}$$

$$a = h \cos x$$
$$h = \frac{a}{\cos x}$$
$$x = \cos^{-1}\left(\frac{a}{h}\right)$$



$$\tan x = \frac{o}{a} \quad \text{(toa)}$$

$$o = a \tan x$$
$$a = \frac{o}{\tan x}$$
$$x = \tan^{-1}\left(\frac{o}{a}\right)$$

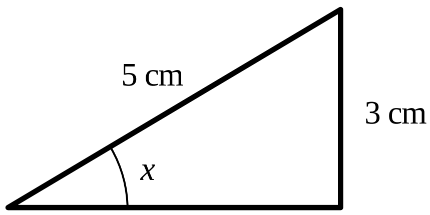
Sine, cosine and tangent (SIN, COS and TAN buttons on calculator) are functions that convert an **angle** into a **ratio of side lengths** in a right angled triangle. They are also very useful in science and engineering in topics such as optics, electricity and vibration.

You should know:

- $\sin(0) = 0$, $\sin(30°) = \frac{1}{2}$, $\sin(90°) = 1$
- $\cos(0) = 1$, $\cos(60°) = \frac{1}{2}$, $\cos(90°) = 0$
- $\tan(0) = 0$, $\tan(45°) = 1$

Finding the angle

The inverse functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ convert the ratio back into the angle.



Here I know all 3 sides, so I can use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ and they will *all give me the same result*!

- $x = \sin^{-1}\left(\frac{o}{h}\right) = \sin^{-1}\left(\frac{3}{5}\right) = 36.87°$
- $x = \cos^{-1}\left(\frac{a}{h}\right) = \cos^{-1}\left(\frac{4}{5}\right) = 36.87°$
- $x = \tan^{-1}\left(\frac{o}{a}\right) = \tan^{-1}\left(\frac{3}{4}\right) = 36.87°$

In general a question will give you just two sides – think "oh", "ah" or "oa" and use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ as appropriate.

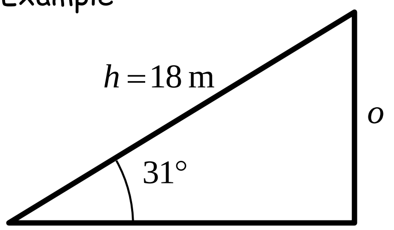
Remember to check that your calculator is in degrees mode (little "**D**" at top of display). At A-level you will learn about radians (a different angle unit, 2π radians = 360°).

Finding a length

- To find a length "on top" in the definition, multiply by the trigonometric function.
- To find a length "underneath" in the definition, divide by the trigonometric function.

You should always write out the formula you are using.

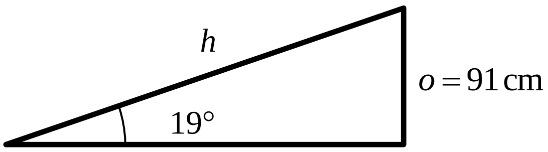
Example



$$\frac{o}{h} = \sin x$$

$$o = h \times \sin(x) = 18 \times \sin(31°) = 9.27\text{ m}$$

Shorter than the hypotenuse? Sensible? That's OK then!



$$\frac{o}{h} = \sin(x) \quad \Rightarrow \quad o = h \times \sin(x)$$

or take reciprocals

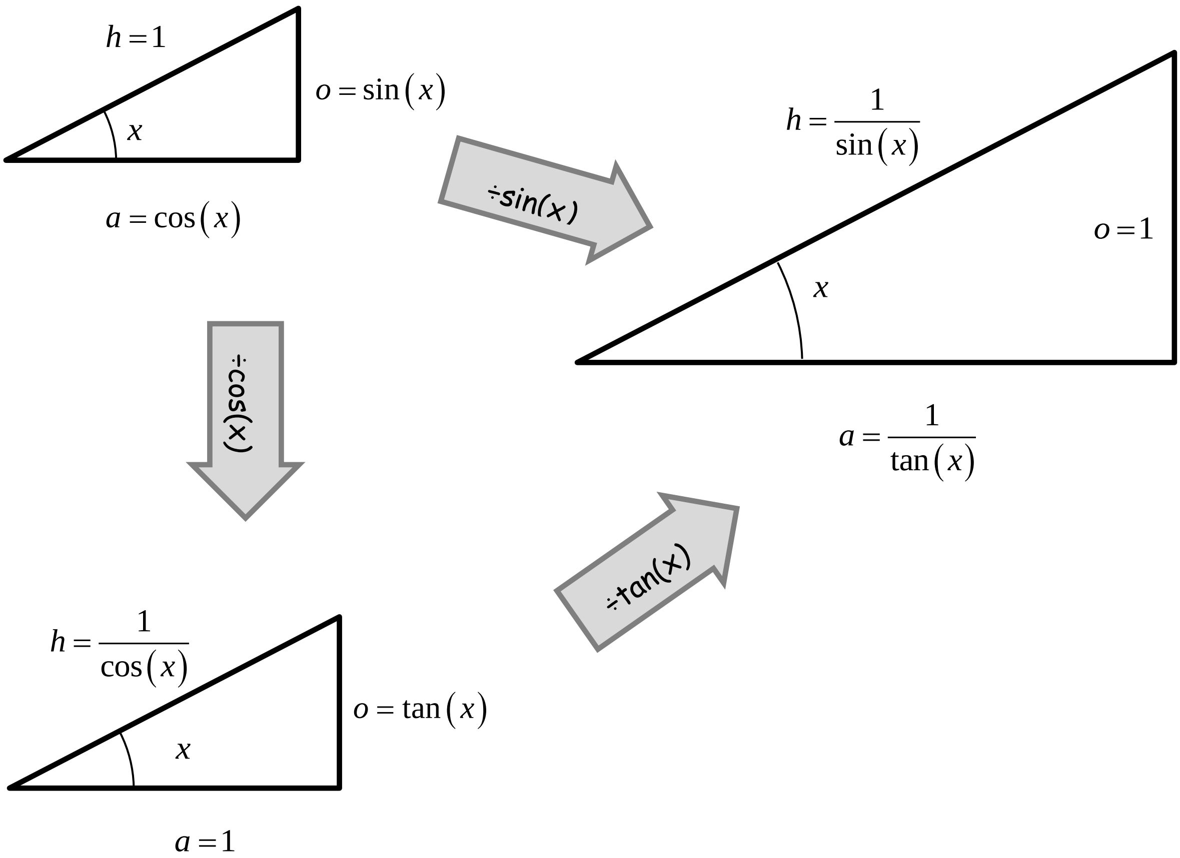
$$\frac{h}{o} = \frac{1}{\sin(x)}$$

$$h = \frac{o}{\sin(x)}$$

$$h = \frac{67\text{ cm}}{\sin(19°)} = 205.8\text{ cm}$$

Longer than *o*? Sensible? That's OK then!

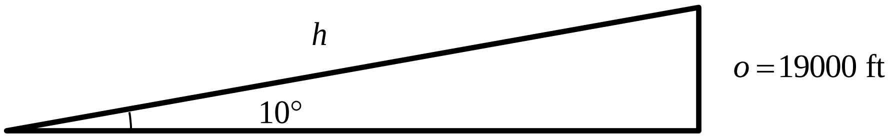
Another way of remembering this: think of your question as a scaled-up version of one of three similar triangles.



Practical problems using trigonometry

(i) as you walk towards Everest base camp, you see the mountain in front of you. The ***angle of elevation*** of the summit is 10° above the horizontal and you know it is 19000 feet above you. How far are you from the summit?

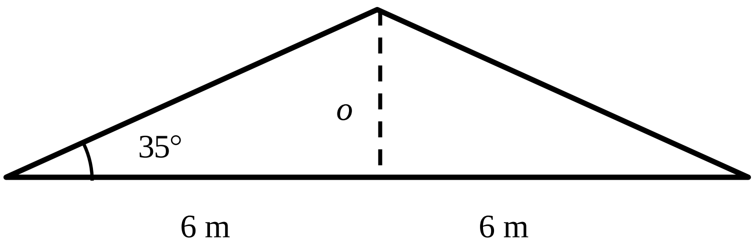
<u>Imagine</u> a right-angled triangle and sketch the problem:



$$h = \frac{o}{\sin(x)} = \frac{19000}{\sin(10)} = 109417 \text{ feet} = \frac{109417}{5280} \text{ miles} = 20.7 \text{ miles}$$

(ii) A symmetrical roof is 12 m wide and the pitch is 35°. How high is it?

We have to draw an isosceles triangle and split it into two right-angled triangles:

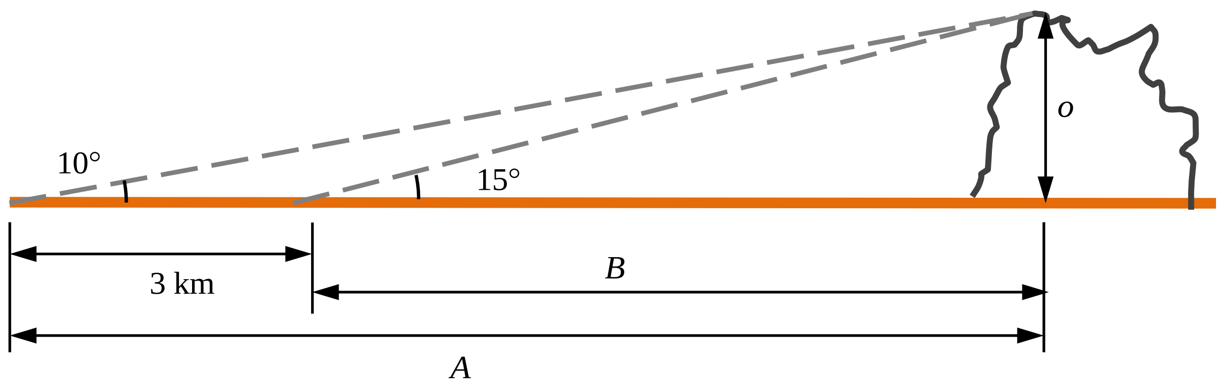


$$\frac{o}{6} = \tan(35°), \quad \text{height} = o = 6 \times \tan(35°) = 4.2 \text{ m}$$

(iii) Challenge question

An explorer is crossing a desert. At 9 am he sees mountains at the edge of the desert; their angle of elevation is 10°. By 10 am he has walked another 3 km and the mountains look closer – the angle of elevation is 15°.

- How high are the mountains (relative to the desert).
- His route takes him in a gorge through the mountains, so he is not walking uphill. At what time will he get there?



Tips:

- Write a relationship between *A* and *o*.
- Write a relationship between *B* and *o*.
- Use the 3 km to combine these into a single equation for *o*.
- Find the mountain height "*o*".
- Find *B*.

$$A = \frac{o}{\tan(10°)}$$

$$B = \frac{o}{\tan(15°)}$$

$$A - B = 3 \text{ km}$$

$$A - b = \frac{o}{\tan(10)} - \frac{o}{\tan(15)} = 3$$

Factorise:

$$o\left(\frac{1}{\tan(10)}-\frac{1}{\tan(15)}\right)=3$$

Putting numbers into this: $1.939o=3$, height of mountain $o=1.55\text{ km}$

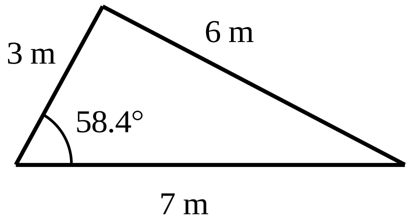
Then using $B=\frac{o}{\tan(15°)}$ we get *B* = 5.77 km. He is walking at 3 km/hour so this takes another $\frac{5.77}{3}=1.92\text{ hours}$ = 1 hour + 0.92×60 minutes = 1 hour 55 minutes after 10 am.

- He gets there at 11:55.

Area of a triangle

The area of any triangle = $\frac{1}{2}ab\sin C$

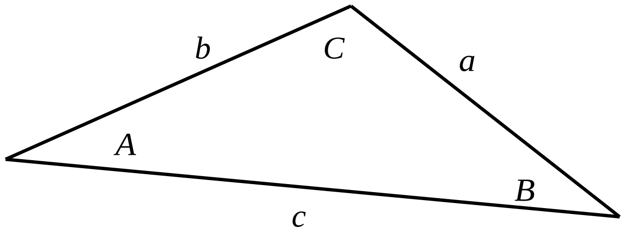
where *C* is the angle between the two sides of length *a* and *b*.



Area = $\frac{1}{2}\times 3\times 7\times\cos(58.4°)=5.5\text{ m}^2$

Sine and cosine rules

The sine and cosine rules are used to calculate lengths or angles in triangles that are not right-angled.



Sine Rule:	$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
Cosine Rule: (etc)	$a^2 = b^2 + c^2 - 2bc\cos A$

$\frac{1}{2}ab\sin C$

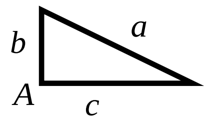
Cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$

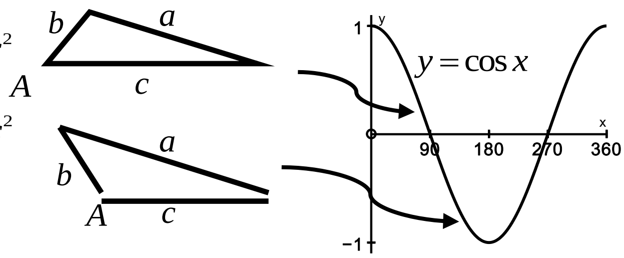
$\therefore \quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Note:

- If $A = 90°$, $\cos A = 0$ and we get $a^2 = b^2 + c^2$ (Pythagoras)

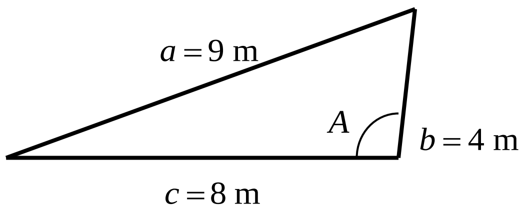


- If $A < 90°$, $\cos A > 0$ and we get $a^2 < b^2 + c^2$
- If $A > 90°$, $\cos A < 0$ and we get $a^2 > b^2 + c^2$



Example

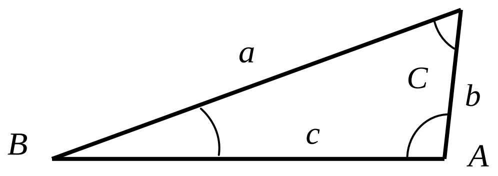
A room with sides of length 4 m and 8 m measures 9 m across the diagonal.
What is the angle opposite the diagonal?



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 16 - 81}{2 \times 8 \times 9} = -0.015625$$

Angle $A = \cos^{-1}\left(-0.15625\right) = 90.895°$

Sine rule



Finding angles: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Finding lengths: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

<u>Example</u>

(Continuing from above) with a = 9 m, b = 4 m, c = 8 m.

What is the angle *B*?

We could use the cosine rule again but it is easier to use the sine rule.

$$\frac{\sin B}{8} = \frac{\sin 90.895}{9}$$

$$\sin B = 8 \times \sin(90.895) \div 9 = 0.8888$$

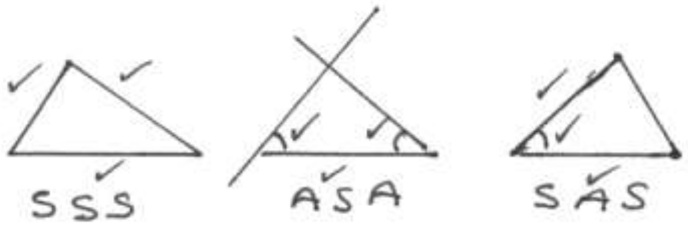
$$B = \sin^{-1}(0.8888) = 62.72°$$

The ambiguous case.

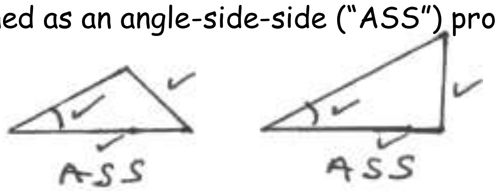
If a triangle is defined in terms of

- 3 sides ("SSS")
- Angle-side-angle ("ASA")
- Side-angle-side ("SAS")

there is only one way it can be drawn.

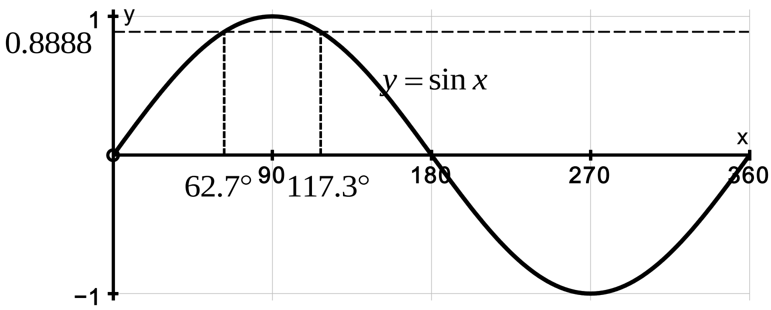


If however it is defined as an angle-side-side ("ASS") problem, it can often be drawn in two ways.



- the sine rule will produce two solutions for the unknown angle
- the cosine rule will produce two solutions for the length of the third side

In general there are two angles <180° with the same sine. In the above example with $\sin B = 0.8888$ we could have taken $B = 62.72°$ or $B = 180 - 62.7 = 117.3°$ but the higher angle is <u>obviously impossible here</u> as it would make the angles in the triangle add to >180°.



Often, though, the angle we know is <90° and we then have two equally valid choices for *B*. (If we knew *C* we could tell which was correct – but then we would not need the sine rule at all!)

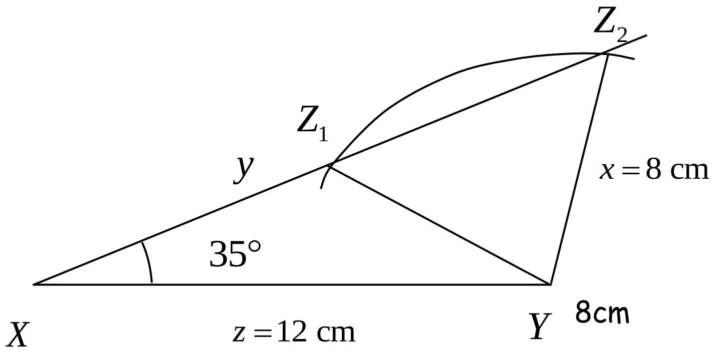
- With ASS problems we usually avoid the cosine rule (which would involve solving a quadratic) and instead use the sine rule to find one unknown angle.
- Since the 3 angles add to 180° we can then find the third angle.

Example.

In triangle XYZ,

- XY = 12cm,
- YZ = 8cm
- angle YXZ = 35°.

Find the size of the angle XZY, the third angle and the length XZ.



Using $\frac{\sin Z}{z} = \frac{\sin X}{x}$ since we know length *z* and one angle and opposite side pair (*X*, *x*)

$$\frac{\sin Z}{12} = \frac{\sin 35°}{8} \quad \text{so} \quad \sin Z = \frac{12 \sin 35^o}{8} = 0.8604$$

Z = 59.4° or 120.6°

- If one possible solution is θ then the other is always 180 - θ. You have to decide whether this is possible or not. (All sine rule calculations of an angle are for an ambiguous "ASS" triangle definition since one of the two known sides must be opposite the known angle).

- Now $Y = 180 - X - Z$ so $Y_1 = 180 - 35 - 120.6 = 24.4°$ and $Y_2 = 180 - 35 - 59.4 = 85.6°$
  - We can now use the sine rule to find length y, e.g. $\frac{y}{\sin Y} = \frac{8}{\sin 35}$ giving $y_1 = 5.76$, $y_2 = 13.91$
  - or the cosine rule $y^2 = 12^2 + 8^2 - 2 \times 12 \times 8 \cos Y$

Which rule to use?

- Use the Sine Rule when you want to end up with 2 lengths and 2 angles
- Use the Cosine Rule when you want to end up with 3 lengths and 1 angle
- Use either when you will be ending up with 3 lengths and 2 angles.