Quadratic Equations

Quadratic equations contain terms where the unknown parameter (typically x) is squared, for instance

- $x^2 - 4x + 3 = 0$
- $5x^2 + 7x + 2 = 0$
- $(x-2)^2 - 3 = 0$

It is absolutely essential that you always recognise a quadratic equation when you see it. There are 3 ways of solving a quadratic – if you do not recognise it and use one of these you will find it totally impossible to solve.

What is "solving"?

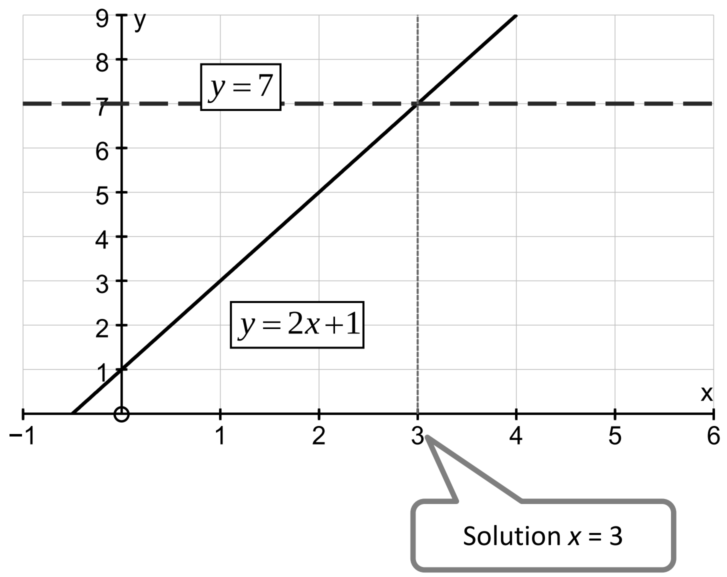
The "solution" of an equation is a number (or numbers) that, when substituted in the equation, make the "=" sign true.

For instance, solving $2x+1=7$ (a **linear** equation), I can try several x values:

x	1	2	3	4
$2x+1$	3	5	**7**	9

Oh look! I wanted 2x+1 to be 7 and when x=3, 2x+1 is actually 7, so $x=3$ is the "solution". (Of course, normally you would use algebra: $2x = 7-1 = 6$, $x = \frac{6}{2} = 3$).

We could think of this as lines on a graph. Where line $y = 2x+1$ crosses line $y=7$, then $2x+1=7$. The x-coordinate of this point is the solution:

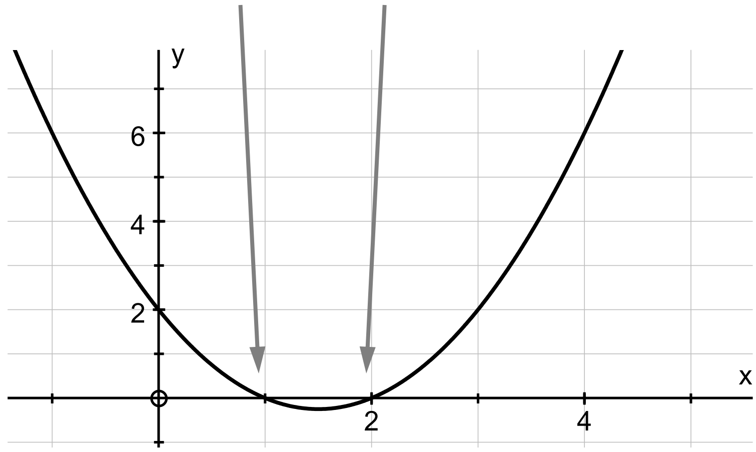


Similarly for a quadratic equation, the solutions are values of x that make it "true".

e.g. $x^2 - 3x + 2 = 0$

x	-1	0	1	2	3	4
$x^2 - 3x + 2$	6	2	0	0	2	6

Now there are two values (x=1, x=2) that "work" in the sense that they result in $x^2 - 3x + 2$ being zero. A quadratic equation typically has 2 roots. As a graph, we are looking for where the curve $y = x^2 - 3x + 2$ crosses the $y = 0$ line (x-axis); we can see the "roots" (another word for solutions) are $x = 1$ and $x = 2$.



Wherever possible, we solve using factorisation. This can also be used for higher order equations (cubic, quartic) in simple cases where we can spot the factors.

Using factorisation to solve equations.

Factorisation can be used to solve equations that can be written as $expression = 0$

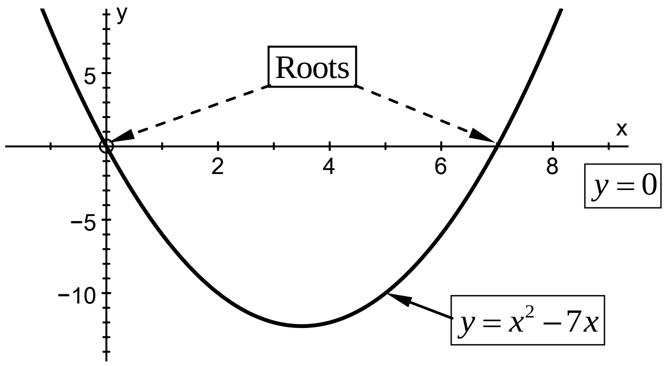
If the expression factorises, we get a number of separate equations of the form $factor = 0$, each of which gives one solution to the problem. This is because if two numbers (first bracket, second bracket) multiply to make zero, one or other of them must be zero (like the "nought times table").

Example. Solve the equation: $x^2 - 7x = 0$

$$\therefore x(x-7) = 0$$

$\therefore$ either x = 0 or x - 7 = 0

x = 7



- If you have forgotten how to factorise, read the unit 2 factorisation notes.

Example. Solve the equation $x^2 - 3x + 2 = 0$

ac=1×2=2. Factors of 2 that add to make -3 are -1, -2 so the factorised form is

$$(x-1)(x-2) = 0$$

Hence either $(x-1) = 0$, which means $x = 1$, or
$(x-2) = 0$, which means $x = 2$. (See graph on previous page).

Example. Solve the equation: $2x^2 - 5x - 12 = 0$

ac = -24 = -8×3

$$\left(2x + \frac{3}{1}\right)\left(1x - \frac{8}{2}\right) = 0$$

(2x + 3)(x - 4) = 0

either: 2x + 3 = 0 or x - 4 = 0

2x = - 3 x = 4

x = - 1.5

If an equation has more than one power of x, it can probably be re-arranged in a quadratic form

Example

Solve the equation $x^2 - x = 30$

Put all the terms on the left hand side so we have an "= 0" form:

$x^2 - x - 30 = 0$

$ac = -30 = -6 \times 5$

$\therefore \quad (x-6)(x+5) = 0$, roots are $x = -5$ and $x = 6$.

Example. Solve the equation: $x + \frac{4}{x} = 5$

- We have integer powers x^1, x^0 and x^{-1}; the range of powers is 2, so it is a quadratic.

Multiply through by x: $x^2 + 4 = 5x$

$x^2 - 5x + 4 = 0$

Factorising: $(x-1)(x-4) = 0$

$\therefore$ either $x - 1 = 0$ or $x - 4 = 0$

$x = 1$ or $x = 4$

Example

Solve the equation $\frac{5x}{x+6} = x - 6$

We "do not like" having a fraction with x in the denominator in any kind of equation, so multiply through by (x+6):

$5x = (x-6)(x+6) = x^2 - 36$

Write in "=0" form: $x^2-5x-36=0$

Factorise: $ac=-36=-9\times 4$ (chosen because $-9+4=-5$).

$$(x-9)(x+4)=0,$$

$$x=9 \text{ or } x=-4.$$

Completing the square of a quadratic function.

Theory

If we can write a quadratic equation in the form $(x+m)^2=n$ then we easily see that $x+m=\pm\sqrt{n}$ and the two roots are $x=-m\pm\sqrt{n}$

Remembering that $(x+a)^2=x^2+2ax+a^2$, we see $x^2+2ax=(x+a)^2-a^2$ so we could write our quadratic as

- $x^2+bx+c=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$
- $ax^2+bx+c=a\left[x^2+\frac{b}{a}x\right]+c=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right]+c$

You do not need to remember the formula itself – just remember the method and follow the laws of algebra.

"Sanity check" – the one you remember.

$$x^2-6x=(x-3)^2-9$$

halve   square and subtract

Example. Solve $x^2 - 6x - 5 = 0$ by "completing the square".

$$x^2-6x-5=\left[(x-3)^2-9\right]-5$$

Always put the first 2 terms in **BIG BRACKETS**

$x^2 - 6x - 5 = (x-3)^2 - 14 = 0$

$(x-3)^2 = 14$

Now the left side is a perfect square, so we can easily square root it:

$(x-3) = \pm\sqrt{14}$ Don't forget the $\pm$, this is what gives you **two roots**.

Hence either $x = 3 + \sqrt{14}$

or $x = 3 - \sqrt{14}$

Example. Express $x^2 - 3x - 5$ in completed square form.

$$x^2 - 3x - 5 = \left(x - \frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 - 5$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 5 = \left(x - \frac{3}{2}\right)^2 - \frac{29}{4}$$

Now to solve $x^2 - 3x - 5 = 0$, we write $\left(x - \frac{3}{2}\right)^2 - \frac{29}{4} = 0$

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

$$\left(x - \frac{3}{2}\right) = \pm\sqrt{\frac{29}{4}}$$

$x = \frac{3}{2} + \frac{\sqrt{29}}{2}$ or $x = \frac{3}{2} - \frac{\sqrt{29}}{2}$

When you have a multiple of x^2, put the multiple **outside the great big brackets**.

The "c" can go inside the brackets as $\frac{c}{a}$ or outside as c – your choice!

Example. Express $15-8x-x^2$ in completed square form

$$15-8x-x^2=-\left[x^2+8x-15\right]$$

$$=-\left[(x+4)^2-4^2-15\right]$$

$$=-\left[(x+4)^2-31\right]$$

$$=31-(x+4)^2$$

Example. Express $2x^2+3x-5$ in the form $A(x + B)^2 + C$.

$$2x^2+3x-5=2\left[x^2+\frac{3}{2}x\right]-5$$

$$=2\left[\left(x+\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]-5=2\left(x+\frac{3}{4}\right)^2-\frac{2\times 9}{16}-5$$

$$=2\left(x+\frac{3}{4}\right)^2-\frac{9}{8}-5=2\left(x+\frac{3}{4}\right)^2-6\frac{1}{8}$$

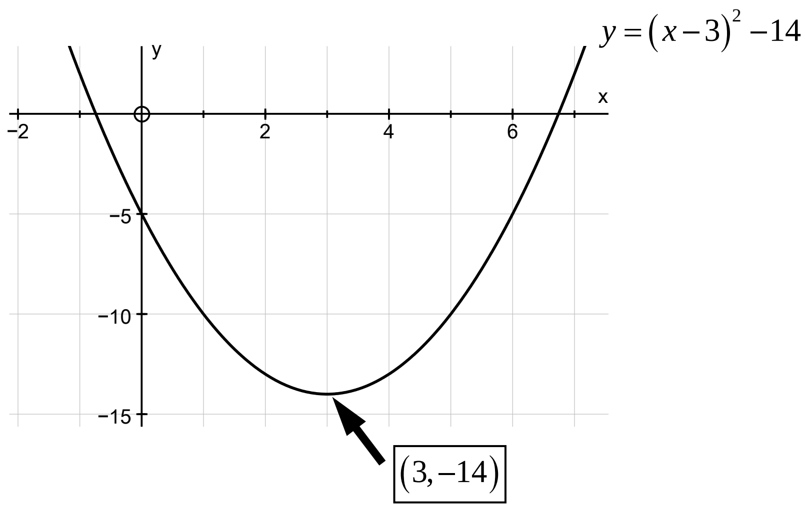
A = 2, B = $\frac{3}{4}$ and C = $-\frac{49}{8}$

Finding the minimum point on a curve.

Once we have completed the square, the minimum point (or maximum if upside down) on the curve can just be spotted.

$y = x^2 - 6x - 5 = (x-3)^2 - 14$

We can make $(x-3)^2$ = zero by setting $x = 3$. Then $y = 0 - 14 = -14$ and the minimum point is $(3, -14)$.



The Quadratic Formula.

A quadratic equation can be written in the general form: $ax^2 + bx + c = 0$, where the "coefficients" *a*, *b* and *c* are constants.

The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should really only be used when the function cannot be factorised as there may be "method marks" for the factorising. Factorising also lets you simplify algebraic fractions – the formula does not help with this.

Example

Solve the equation $5x^2 + 2x - 3 = 0$

The coefficients are: a = 5, b = 2, c = -3

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 5 \times (-3)}}{2 \times 5}$$

$$= \frac{-2 \pm \sqrt{4 + 60}}{10} = \frac{-2 \pm 8}{10} = -1 \text{ or } +\frac{6}{10}$$

Quadratics in disguise

Quadratic questions may be made more difficult (more marks!) by expecting you to do some algebra before you start solving the equation.

Remember the "rules of algebra". You are trying to make an equation equivalent to the original (i.e. with the same solutions) – but easier to solve. Whatever you do to one side of the equation, you must do to the other side, otherwise you have corrupted the equation and it will not have the same solutions (*Oh dear!*).

You are allowed to:

- add or subtract something
- multiply by something, divide by something
- take the reciprocal of <u>the whole of</u> each side

The question may want you to:

- start with an equation like $x^2 = 2x + 3$ and move all the terms to one side so it "looks like" a quadratic: $x^2 - 2x - 3 = 0$
- manipulate fractions like $x = 2 + \frac{3}{x}$ into a form like $x^2 = 2x + 3$, then $x^2 - 2x - 3 = 0$
- start with 4 terms like $x^2 - ax + bx - ab$ instead of the usual 3 terms
- Write an equation for lines intersecting
- write an equation for a "word equation"

Examples

(a) **Collecting all the terms on one side**

$x^2 = 2x + 3$

I want to keep the x^2 on the left side, so I will subtract $2x + 3$ from each side, to make the right hand side (RHS) = 0

$x^2 - 2x - 3 = 0$

This is just a quadratic – factorise and solve.

(b) **Multiplying out and collecting terms**

$(x+2)(2x+3)=(x+1)(x-3)$ There are no common factors so I must multiply out:

$$2x^2+3x+4x+6=x^2-3x+x-3$$

$$2x^2+7x+6=x^2-2x-3$$

Now to get all the terms on one side I take x^2 from each side:

$$x^2+7x+6=-2x-3$$

Next, add $2x+3$ to each side: $x^2+9x+9=0$. Hey, we've got a quadratic! (solve using formula).

(c) **How to get rid of fractions.**

(i) Any equation you get with a fraction will turn into either a quadratic or a simple linear equation. Just multiply every term by the denominator you don't like.

$$x=2+\frac{3}{x}$$

I can get rid of the $\frac{\ }{x}$ fraction by multiplying it by x. Of course, I have to multiply all the other terms by x too.

Remember how to multiply a fraction by a number:

$\frac{1}{4}\times 3=\frac{1\times 3}{4}=\frac{3}{4}$ (not $\frac{3}{12}$, that's just the same as $\frac{1}{4}$!!)

$$x^2=2x+\frac{3\cancel{x}}{\cancel{x}}=2x+3$$

Now I just move all the terms to one side: $x^2-2x-3=0$ and I have a quadratic.

(ii) With several fractions, I multiply by each denominator in turn until there are none left:

$\frac{1}{x}+\frac{3}{x+1}=1$, multiply by x:

$\frac{x}{x}+\frac{3x}{x+1}=x$ hence $1+\frac{3x}{x+1}=x$

Now multiply by $x+1$

$(x+1)+\frac{3x(x+1)}{(x+1)}=x(x+1)$, simplify as $(x+1)+3x=x^2+x$, then $4x+1=x^2+x$

Collect terms on one side: $x^2-3x-1=0$, *quadratic equation!*

(iii) with **just one** fraction each side you can take reciprocals:

$\frac{1}{x^2}=\frac{1}{2x+3}$ becomes $x^2=2x+3$

(iv) Conversely $\frac{1}{x^2}=\frac{1}{2x}+\frac{1}{3}$ has 2 fractions on the right side so all we can do is multiply out. Just multiply by $6x^2$:

$\frac{6x^2}{x^2}=\frac{6x^2}{2x}+\frac{6x^2}{3}$, simplify: $6=3x+2x^2$

Now put into =0 form: $2x^2+3x-6=0$

(d) **Equations that start with four terms**

$x^2-ax+bx-ab=0$

This is already half-way to being factorised! Just factorise the first pair of terms, factorise the second and combine:

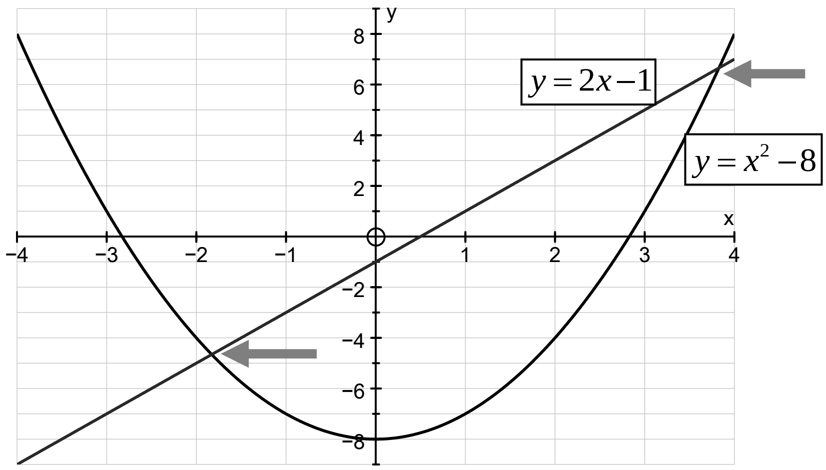
$x^2-ax+bx-ab=x(x-a)+b(x-a)=(x+b)(x-a)=0$

Brackets must be the same

Solutions $x=a$, $x=-b$

Equations for intersecting lines

We saw in "Equations of curves" that the point where two lines intersect (cross each other) is the solution of a pair of equations.



Now you know how to actually solve the equations (as opposed to estimating the x-value from the graph). Where they cross, they have the same x and y-coordinates

$y = x^2 - 8$ and $y = 2x - 1$

Hence $x^2 - 8 = 2x - 1$

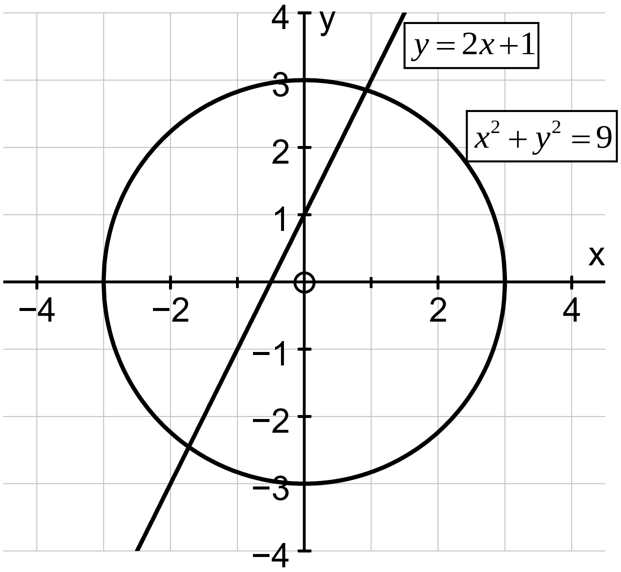
$x^2 - 2x - 8 + 1 = 0$

$x^2 - 2x - 7 = 0$ Quadratic equation !

We can do the same with the equation of a circle

"Find the points of intersection of the line $y = 2x + 1$ and a circle, radius 3, centered on the origin".

The equation of the circle is $x^2 + y^2 = 3^2$



Substituting $2x+1$ in place of y:

$x^2 + (2x+1)^2 = 9$

$x^2 + 4x^2 + 4x + 1 = 9$

$5x^2 + 4x - 8 = 0$ *Quadratic equation !*