<u>Factorising quadratics</u>

A <u>quadratic</u> expression is one containing a letter squared e.g.

$x^2 + 4x + 3$ or $3y^2 - 2y - 1$

All quadratics can be written as $ax^2 + bx + c$ where a, b, c are numbers ("coefficients").
Sometimes they have other letters, e.g. **y**, **z** etc instead of x.

Equations containing x^2 ("quadratic equations") can often be solved by factorising (unit 3).
This is a key skill both for gcse and A-level, one of the things that is essential for an A-grade.

<u>Simple quadratics (a=1), just x^2, not $3x^2$</u>

"The opposite of multiplying two brackets"

We know $(x + 3)(x+4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$.

How do we reverse the process to factorise $x^2 + 7x + 12$?

- ➔ need (x + number)(x + another number)
- ➔ think $(x + b)(x + a) = x^2 + ax + bx + ab$

$$= x^2 + (a+b)x + ab$$

<u>To find the numbers for $x^2 + 7x + 12$, look for two numbers a, b that are factors of 12 and add to 7.</u>

Factors of 12: 1×12, 2×6, 3×4, also -1×-12 etc

Factors add to 1+12=13, 2+6=8, **3+4=7**, -1-12=-13

Use the 3 and 4: (x + 3)(x+4) ...order does not matter.

Now check it works! $(x + 3)(x+4) =x^2 + 4x + 3x + 12 = x^2 + 7x + 12$

The key skill here is <u>finding number pairs</u>:

Product 24, sum 10?

To make +24, do we want "two positive factors" 1×24, 2×12, 3×8, 4×6

or "two negative" -1×-24, -2×-12, -3×-8, -4×-6

- ➔ 4,6 is the right pair since 4+6 = 10.

Practice:

Product 32, sum 12 (4,8)
Product 15, sum 16 (1,15)
Product 15, sum -16 (-1, -15)
Product 18, sum -11 (-2, -9)
Product -10, sum 9 (-1, 10)
Product -10, sum -3 (-5, 2)

If the "c" term is positive, numbers have the same sign (++ or --):
Two positive numbers – **multiply these out and check they work**!

$x^2 + 7x + 10 = (x+2)(x+5)$

$x^2 + 11x + 10 = (x+1)(x+10)$

$x^2 + 31x + 30 = (x+1)(x+30)$

$x^2 + 17x + 30 = (x+2)(x+15)$

$x^2 + 13x + 30 = (x+3)(x+10)$

$x^2 + 11x + 30 = (x+5)(x+6)$

two negative:

$x^2 - 7x + 10 = (x-2)(x-5)$

$x^2 - 11x + 10 = (x-1)(x-10)$

$x^2 - 31x + 30 = (x-1)(x-30)$

If "c" is negative, need + and –

$x^2 + 9x – 10 = (x+10)(x-1)$

$x^2 + 3x – 10 = (x+5)(x-2)$

$x^2 - 3x -10 = (x+2)(x-5)$

$x^2 - 9x -10 = (x+1)(x-10)$

Factorising the "difference of two squares"

$(x + a)(x – a) = x^2 –ax + ax – a^2 = x^2 – a^2$

(remember $(x+a)^2 = (x+a)(x+a) =$

and $(x-a)^2 = ...$)

We call anything like $x^2 – a^2$ a "difference of 2 squares",

- Just write the factors as $(x+a)(x-a)$

e.g. $x^2 – 5^2 = (x+5)(x-5)$

Alternatively, write it as $x^2 + 0x – 25$ and look for factors of -25 that add to 0 (need +5 and -5) hence $(x+5)(x-5)$.

Examples:

$x^2 – 1 = (x+1)(x-1)$

$100 – p^2 = (10 + p)(10 – p)$

$11^2 - 9^2 = (11 + 9)(11 – 9) = 20 \times 2 = 40$

$(3x)^2 – (2y)^2 = (3x + 2y)(3x – 2y)$

$(x+2)^2 – (y-5)^2 = ((x+2) + (y-5)) ((x+2) – (y-5))$

You may need to divide by a common factor first to get a square number form e.g.

$5x^2 – 500 = 5(x^2 – 100) = 5(x + 10)(x – 10)$

$2x^2- 8 = 2(x^2 – 4) = 2(x+2)(x-2)$

Factorising expressions like $2x^2 + 5x + 2$

Think of "$ax^2 + bx + c$"

Method:

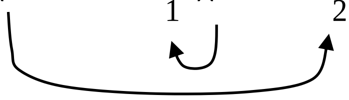
1. Find ac = 2×2=4
2. Find a pair of factors of ac that add to b=5, do I need 1×4, 2×2, -1×-4, -2×-2 ?
   1×4=ac=4 and 1+4=5=b

3. Because these are factors of **ac** not **c**, we need to divide them by factors of **a** when putting them into the brackets. We do it like this:
   Pick a pair of factors of $2x^2$ such as 2x × 1x, put them in the brackets:

   (2x)(1x)

   Pair the 2 and 1 with a fraction in the other bracket

   $(2x \quad \frac{}{1})(1x \quad \frac{}{2})$

   Put the 1 and 4 into the brackets. Either way around will work – try to make integers.

   $\left(2x+\frac{1}{1}\right)\left(1x+\frac{4}{2}\right)=(2x+1)(x+2)$. Now check it!

   Examples

   $2x^2+3x+1$, ac = 2 = 1×2, $2x^2+3x+1=\left(2x+\frac{1}{1}\right)\left(1x+\frac{2}{2}\right)=(2x+1)(x+1)$

   $3x^2+7x+2$, ac = 6 = 1×6, $3x^2+7x+2=\left(3x+\frac{1}{1}\right)\left(1x+\frac{6}{3}\right)=(3x+1)(x+2)$

   $6x^2-13x+6$, ac = 36 = -4×-9, shall we say $6x^2$ = 1x×6x or 2x×3x ?

   $$6x^2-13x+6=\left(3x-\frac{4}{2}\right)\left(2x-\frac{9}{3}\right)=(3x-2)(2x-3)$$

   If you did $6x^2-13x+6=\left(6x-\frac{4}{1}\right)\left(1x-\frac{9}{6}\right)=(6x-4)\left(x-\frac{3}{2}\right)$

   you are not wrong but you need to finish by halving the first bracket and doubling the second to make integers:

   $(6x-4)\left(x-\frac{3}{2}\right)=(3x-2)(2x-3)$

Factorising 4 term expressions.

An alternative to putting fractions in the brackets is to split the 3 quadratic terms into 4 terms.

- The bx term splits in two, using the factors of ac
- Factorise the first pair, factorise the second pair
- Then combine the brackets

Examples:

(a) $2x^2+3x+1$, $ac = 2 = 1\times2$

Hence we can write $2x^2+3x+1 = 2x^2+1x + 2x+1$

Factorise the pairs: $= x(2x + 1) + 1(2x + 1)$

Combine them: $= (x + 1)(2x + 1)$

(b) $6x^2-13x+6$, $ac = 36 = -9 \times -4$

Hence we can write $6x^2-13x+6 = 6x^2-9x -4x+6$

Factorise the pairs: $=3x(2x - 3) - 2(2x - 3)$ [n.b. not $+2(-2x + 3)$, the two brackets <u>must be identical</u>]

Combine: $= (3x - 2)(2x - 3)$

You may be asked to factorise a 4 term expression that is <u>not a quadratic</u>.

- Factorise pairs of terms, them combine, as above

Example:

$xy + 4x + 3y + 12$

$= x(y + 4) + 3(y + 4)$

$= (x + 3)(y + 4)$

Proof of the fractions method (not needed for gcse!)

$$\left(px+\frac{r}{q}\right)\left(qx+\frac{s}{p}\right)=pqx^2+(r+s)x+\frac{rs}{pq}=ax^2+bx+c$$

r,s are the two factors of $ac=pq\left(\frac{rs}{pq}\right)=rs$ that add to $r+s=b$

We also need pq = a.