Vector closest point questions

1.(a) $r = \begin{pmatrix}0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix} = \underline{a} + \lambda\underline{b}$

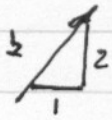
Need $\underline{r}\cdot\underline{b} = 0$, $b = \begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{pmatrix}0\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = 6 + 5\lambda = 0$$

$$\lambda = -6/5 = -1.2$$

(b) At $\lambda = -1.2$, $\underline{r} = \begin{pmatrix}-1.2\\0.6\end{pmatrix}$

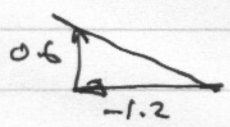
$$|r| = \sqrt{1.2^2 + 0.6^2} = 0.6\sqrt{2^2+1^2} = 0.6\sqrt{5}$$

(c)(i) 

gradient 2, y-intercept (0,3)

$y = 2x + 3$

(ii) P is $\begin{pmatrix}-1.2\\0.6\end{pmatrix}$



$m = -1/2$

$y = -1/2\,x$.

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2. $r = \begin{pmatrix}10\\-6\\7\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\4\end{pmatrix}, = \underline{a} + \lambda\underline{b}$,

need to find λ such that $\underline{r}\cdot\underline{b} = 0$

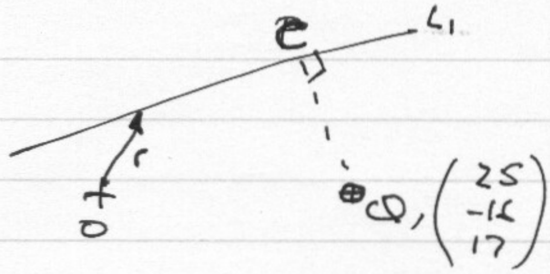
$$\begin{pmatrix}10\\-6\\7\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\4\end{pmatrix}\begin{pmatrix}5\\-2\\4\end{pmatrix} = (50+12+28) + \lambda(25+4+16)$$

$$= 90 + 45\lambda = 0$$

$$\therefore \lambda = -90/45 = -2$$

$OP = \begin{pmatrix}10-10\\-6+4\\7-8\end{pmatrix} = \begin{pmatrix}0\\-2\\-1\end{pmatrix}$, $|OP| = \sqrt{5}$

3 (a) $r = \begin{pmatrix}10\\-6\\7\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\4\end{pmatrix}$



$\vec{QC} = r - \begin{pmatrix}25\\-16\\17\end{pmatrix}$

$= \begin{pmatrix}-15\\10\\-10\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\4\end{pmatrix}$

$\vec{QC}\cdot\begin{pmatrix}5\\-2\\4\end{pmatrix} = (-75-20-40) + \lambda(25+4+16)$

$= -135 + 45\lambda = 0$

$\therefore \lambda = 135/45 = 3$

(b) $|QC| = \left|\begin{pmatrix}0\\4\\2\end{pmatrix}\right| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

(c) $\vec{QC} = \begin{pmatrix}0\\4\\2\end{pmatrix}$, $r_2 = \vec{OQ} + \mu\,\vec{QC}$

$= \begin{pmatrix}25\\-16\\17\end{pmatrix} + \mu\begin{pmatrix}0\\4\\2\end{pmatrix}$

(can simplify as $r_2 = \begin{pmatrix}25\\-16\\17\end{pmatrix} + \nu\begin{pmatrix}0\\2\\1\end{pmatrix}$).

(d)

$\cos\theta = \dfrac{\vec{OQ}\cdot\vec{QC}}{|OQ|\,|QC|}$

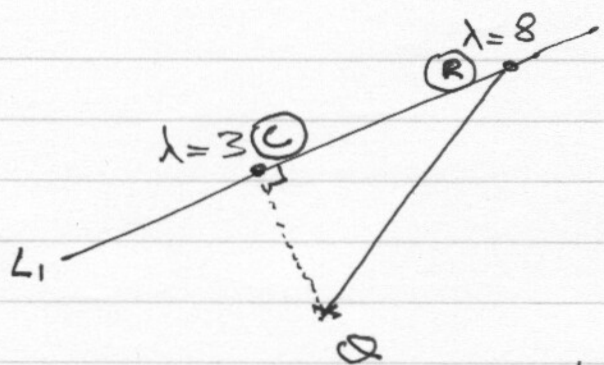
$= \dfrac{\begin{pmatrix}25\\-16\\17\end{pmatrix}\cdot\begin{pmatrix}0\\2\\4\end{pmatrix}}{\sqrt{25^2+16^2+17^2}\,\sqrt{20}} = \dfrac{36}{\sqrt{1170}\,\sqrt{20}}$

$= 0.23534$

$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = 0.97191$

Area = "$\tfrac12 ab\sin C$" = $\tfrac12\sqrt{1170}\,\sqrt{20} \times 0.97191 = 74.34$

3(e)



Distance $|CR| = (8-3)\left|\begin{pmatrix}5\\-2\\4\end{pmatrix}\right| = 5\sqrt{45} = 5(3\sqrt{5})$
$= 15\sqrt{5}$

Since $CQ \perp$ to line L_1, area of triangle CQR

$= \frac{1}{2}|CR|\,|CQ| = \frac{1}{2}(15\sqrt{5})(2\sqrt{5}) = 75$

(f) Point P was on the line at $\lambda = -2$
C is at $\lambda = 3$
$\therefore |PC| = 5\left|\begin{pmatrix}5\\-2\\4\end{pmatrix}\right| = 5\sqrt{45} = 15\sqrt{5}$,
$=$ Same as $|CR|$.

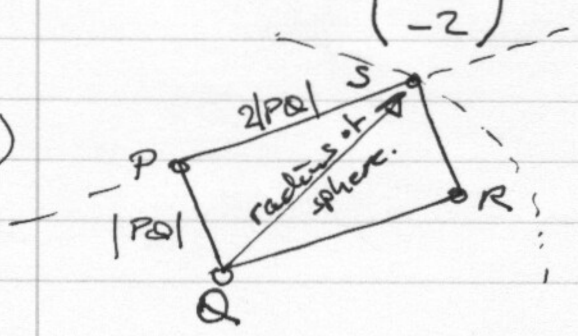
4.(a) $QP = \begin{pmatrix}18-8\\2--3\\3-1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}10\\5\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}$.

Let $\underline{b} = \begin{pmatrix}1\\2\\5\end{pmatrix}$, we need $\vec{QP}.\underline{b} = 0$

$(10+10+10) + \lambda(1+4+25) = 0$
$30 + 30\lambda = 0, \ \lambda = \frac{-30}{30} = -1$

$\therefore P$ is at $\begin{pmatrix}17\\0\\-2\end{pmatrix}$

(b)



radius $r = \sqrt{1^2+2^2}\,|PQ|$
$= \sqrt{5}\,|PQ|$

At $\lambda = -1$,

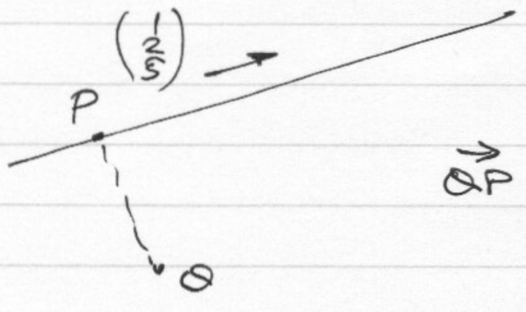
4 (b)+ $|PQ| = \left|\begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}\right| = 3\sqrt{3^2+1^2+1^2} = 3\sqrt{11}$

Surface area of a sphere $= 4\pi r^2$

$= 4\pi\left(\sqrt{5}\times 3\sqrt{11}\right)^2$

$= 4\pi\left(9\times 55\right) = 1980\pi$

(c) We want a vector perpendicular to $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ (the line direction) and also perpendicular to $\vec{QP}$.



$\vec{QP} = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

[Scale it later]

The vector can be any length, so try $\begin{pmatrix} 1 \\ y \\ z \end{pmatrix}$.

Need:

$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} = 0 \quad \therefore \quad 1 + 2y + 5z = 0$

Need: $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} = 0 \therefore \quad 3 + y - z = 0$

(×2) $6 + 2y - 2z = 0$

Subtract: $(1+2y+5z) - (6+2y-2z) = -5 + 7z = 0$

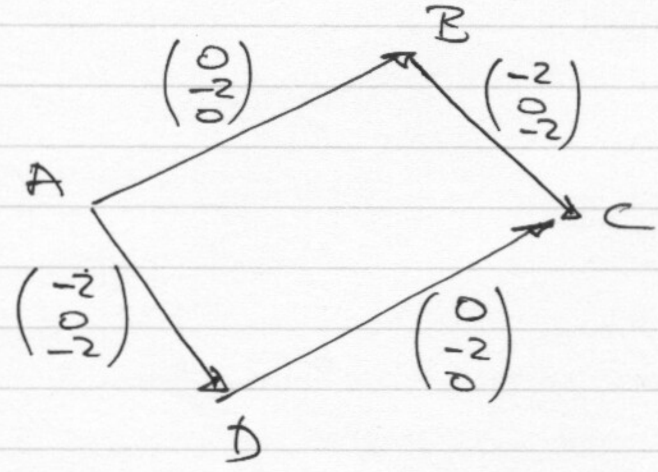
$z = \frac{5}{7}$

$y = z - 3 = \frac{5-21}{7} = -\frac{16}{7}$

Common normal $\underline{n} = \begin{pmatrix} 1 \\ -16/7 \\ 5/7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ -16 \\ 5 \end{pmatrix}$

$\hat{n} = \frac{1}{\sqrt{330}}\begin{pmatrix} 7 \\ -16 \\ 5 \end{pmatrix}$

5.



By inspection, $\vec{AB}$ and $\vec{DC}$ are parallel so points ABC & D are coplanar.

$|AB| = 2, \quad |BC| = \sqrt{8} = 2\sqrt{2}$.

$\vec{AB} \cdot \vec{BC} = 0$ so the angles are 90°.

$\therefore$ It is a rectangle.

6. See C4 notes.