C4 June 2007

1. $$f(x) = (3+2x)^{-3}$$
$$= \left[3\left(1+\tfrac{2}{3}x\right)\right]^{-3}$$
$$= 3^{-3}\left(1+\frac{2x}{3}\right)^{-3}$$

$|x| < 3/2$ so $2/3\, x < 1 \Rightarrow$ can be expressed as a binomial series

$$\left(1+\frac{2x}{3}\right)^{-3} = 1 + (-3)\left(\frac{2x}{3}\right) + \frac{(-3)(-4)}{2}\left(\frac{2x}{3}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{2x}{3}\right)^3 + \dots$$

$$= 1 - 2x + 6\left(\frac{4x^2}{9}\right) - 10\left(\frac{8x^3}{27}\right) + \dots$$

$$= 1 - 2x + \frac{8}{3}x^2 - \frac{80}{27}x^3 + \dots$$

$$\therefore f(x) = \frac{1}{27} - \frac{2}{27}x + \frac{8}{81}x^2 - \frac{80}{729}x^3 + \dots$$

2. Let $u = 2^x$

Need $\frac{dx}{du}$.

$\ln u = \ln(2^x) = x\ln 2$

$\therefore\ x = \left(\frac{1}{\ln 2}\right)\ln u$

$\frac{dx}{du} = \left(\frac{1}{\ln 2}\right)\frac{1}{u}$

At $x=0$, $u=1$

At $x=1$, $u=2$

$$\int_{x=0}^{1} \frac{2^x}{(2^x+1)^2}\,dx = \int_{u=1}^{2} \frac{u}{(u+1)^2}\frac{dx}{du}\,du$$

$$= \int_1^2 \frac{1}{\ln 2\,(u+1)^2}\,du$$

$$= \frac{1}{\ln 2}\int_1^2 (u+1)^{-2}\,du = \frac{1}{\ln 2}\left[-(u+1)^{-1}\right]_1^2$$

$$= \frac{1}{\ln 2}\left[-\frac{1}{3}+\frac{1}{2}\right] = \frac{1}{\ln 2}\left(\frac{1}{6}\right) = \frac{1}{6\ln 2}$$

3.(a) $\int x \cos 2x \, dx$. Use integration by parts since x becomes 1 when differentiated.

$$\int uw \, dx = uW - \int u'W \, dx \qquad \text{where } W = \int w \, dx.$$

Let $x = u$, $w = \cos 2x$

$$\int x \cos 2x \, dx = x\left(\tfrac{1}{2}\sin 2x\right) - \int 1\left(\tfrac{1}{2}\sin 2x\right) dx$$

$$= \tfrac{1}{2} x \sin 2x + \tfrac{1}{4}\cos 2x + c$$

(b) $\cos 2x = 2\cos^2 x - 1 \qquad \therefore \; 2\cos^2 x = \cos 2x + 1$

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2}$$

$$\therefore \int x \cos^2 x \, dx = \tfrac{1}{2}\int x \cos 2x \, dx + \int \tfrac{1}{2} x \, dx$$

$$= \tfrac{1}{4} x \sin 2x + \tfrac{1}{8}\cos 2x + \frac{x^2}{4} + c$$

4.(a) By inspection, $A = 2$ since in terms of x^2 we have $\dfrac{8x^2 + \ldots}{4x^2 + \ldots} = A +$ fractions.

or long division: $(2x+1)(2x-1) = 4x^2 - 1$

$$\begin{array}{r|l} & \;2 \leftarrow A \\ 4x^2 - 1 & \;8x^2 + 0x + 2 \\ & \;\underline{8x^2 \qquad\; - 2} \; - \\ & \qquad\qquad\; 4 \end{array}$$

Then $\dfrac{4}{(2x+1)(2x-1)} = \dfrac{B}{2x+1} + \dfrac{C}{2x-1}$ (converted to a proper fraction)

$\times (2x+1)(2x-1)$ each side:

$$4 = B(2x-1) + C(2x+1).$$

Let $x = 1/2 \rightarrow 4 = 0B + 2C, \; C = 2$

Let $x = -1/2 \rightarrow 4 = -2B + 0C, \; B = -2$

(b) $$\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,dx = \int_1^2 2 - \frac{2}{2x+1} + \frac{2}{2x-1}\,dx$$

$$= \Big[2x - \ln(2x+1) + \ln(2x-1)\Big]_1^2$$

$2x-1$ positive, don't need $|x|$ signs here

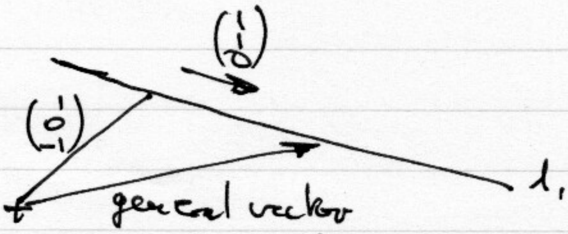
$$= \left[2x + \ln\left(\frac{2x-1}{2x+1}\right)\right]_1^2$$

$$= \left(4+\ln\left(\tfrac{3}{5}\right)\right) - \left(2+\ln\left(\tfrac{1}{3}\right)\right)$$

$$= 2 + \ln\left(\frac{3/5}{1/3}\right) = 2+\ln\left(\tfrac{9}{5}\right)$$

$$= 2+\ln k, \quad k = 9/5.$$

5(a)



$r_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ on line l_1

$r_2 = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$ on line l_2

Solve any two of the i, j, k component equations to find λ and μ, see if compatible with the third equation.

i-components : $1+\lambda = 1+2\mu$

j " : $\lambda = 3+\mu$ subtract

$1 = -2+\mu$, $\mu = 3$, $\lambda = 3+\mu = 6$

k-components : $-1+0\lambda = -1$ on line l_1 } $\therefore$ lines do

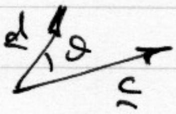
but $6-\mu = 3$ on line l_2 } not meet.

(b) A is at $\begin{pmatrix}1\\0\\-1\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} = \underline{a}$, say.

B is at $\begin{pmatrix}1\\3\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\5\\4\end{pmatrix} = \underline{b}$, say.

Then $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$

For any two vectors $\underline{c}, \underline{d}$



$\underline{c} \cdot \underline{d} = |c||d|\cos\theta, \quad \cos\theta = \dfrac{\underline{c}\cdot\underline{d}}{|c||d|}.$

Here $\underline{c} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \ \underline{d} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ (direction of l_1).

$$\therefore \cos\theta = \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\left|\begin{matrix}3\\4\\5\end{matrix}\right| \times \left|\begin{matrix}1\\1\\0\end{matrix}\right|} = \frac{3+4}{\sqrt{50}\ \sqrt{2}} = \frac{7}{\sqrt{100}} = \frac{7}{10}$$

6. $x = \tan^2 t, \quad y = \sin t, \quad 0 < t < \pi/2$

(a) $dy/dx = dy/dt \times dt/dx = \dfrac{\left(dy/dt\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{\cos t}{2\tan t \sec^2 t}$

$$= \frac{\cos^3 t}{2\tan t}$$

(b) At $t = \pi/4$, $x = (\tan \pi/4)^2 = 1$

$$y = \sin \pi/4 = \frac{\sqrt{2}}{2}$$

$$dy/dx = \frac{(1/\sqrt{2})^3}{2\times 1} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$$

$y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8}(x-1), \quad y = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8}x$

$$= \frac{3}{8}\sqrt{2} + \frac{\sqrt{2}}{8}x.$$

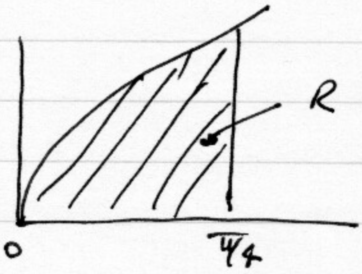
$$= \frac{\sqrt{2}}{8}x + \frac{3}{8}\sqrt{2}, \quad a = \frac{\sqrt{2}}{8},$$

$$b = 3/8\sqrt{2}.$$

c) $x = \tan^2 t = \dfrac{\sin^2 t}{\cos^2 t} = \dfrac{\sin^2 t}{1 - \sin^2 t} = \dfrac{y^2}{1-y^2}$

$x(1-y^2) = y^2, \quad x = y^2 + xy^2 = (1+x)y^2, \quad \boxed{y^2 = \dfrac{x}{1+x}}$

7. (a)



h

x	0	$\pi/16$	$\pi/8$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
$y=$	0	0.44600	0.64359	0.81742	1

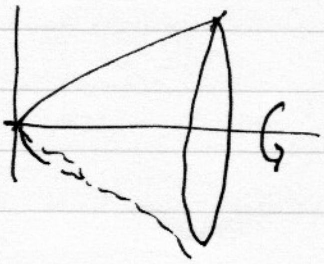
(b) $h = \pi/16$ (= width of strips)

Formula book $\int y\,dx \approx \frac{h}{2}\left(y_0 + y_n + 2(y_1 + y_2 + \dots)\right)$

$= \frac{\pi}{32}\left(0 + 1 + 2(0.446 + .64359 + 0.81742)\right)$

$= 0.4726$ (to 4 d.p.).

(c)



$\text{Volume} = \int_0^{\pi/4} \pi y^2\,dx$

$= \pi \int \left(\sqrt{\tan x}\right)^2 dx = \pi \int_0^{\pi/4} \tan x\,dx$

$= \pi \left[\ln(\sec x)\right]_0^{\pi/4}$ (formula book)

$= \pi\left(\ln(\sqrt{2}) - \ln(1)\right)$

$= \pi \ln(\sqrt{2})$ or $\frac{\pi}{2}\ln 2$

8. (a) $\frac{dP}{dt} = kP$

Integrate wrt t to get rid of the derivative. Must first get all the P terms on the same side.

$\frac{1}{P}\frac{dP}{dt} = k$

$\int \frac{1}{P}\frac{dP}{dt}\,dt = \int \frac{1}{P}\,dP = \int k\,dt$

$\ln(P) = kt + c$

$P = e^{kt+c} = e^c e^{kt} = Ae^{kt}$

where c and A are constants.

At $t=0$, $P = Ae^0 = A$

but $P = P_0 \therefore A = P_0$, $\underline{P = P_0 e^{kt}}$

8 (b) $k = 2.5$

$$P = P_0 e^{2.5t} \quad \text{or} \quad \frac{P}{P_0} = e^{2.5t}$$

If $P = 2P_0$, $\quad \frac{P}{P_0} = 2 = e^{2.5t}$

$2.5t = \ln 2$, $\quad t = \frac{\ln 2}{2.5} = 0.277259$ days

$= 0.277259 \times 24 \times 60$ min

$= 399.3$ minutes

(399 to nearest minute).

(c) $\frac{dP}{dt} = \lambda P \cos \lambda t$

$$\frac{1}{P} \frac{dP}{dt} = \lambda \cos \lambda t$$

$$\int \frac{1}{P} \frac{dP}{dt} dt = \int \frac{1}{P} dP = \int \lambda \cos \lambda t \, dt$$

$$\ln (P) = \sin \lambda t + c$$

$$P = e^{\sin \lambda t} e^c = P_0 e^{\sin \lambda t}$$

(since at $t = 0$, $e^{\sin 0} = 1$).

(d) $P = 2P_0 \quad \therefore e^{\sin \lambda t} = 2$, $\quad \sin \lambda t = \ln 2$

$\lambda t = \sin^{-1}(\ln 2) = 0.76585$ for first time,

→ don't need the repeats

$\therefore t = \frac{0.76585}{2.5} = 0.306338$ days

$= 441.1$ minutes

$= 441$ minutes (to nearest minute).