C4 Jan 2011

1. $\int_0^{\pi/2} x\sin 2x\,dx$ use $\int uW\,dx = uW - \int u'W\,dx$

$= \left[-\frac{x}{2}\cos 2x\right]_0^{\pi/2} - \int_0^{\pi/2} -\frac{1}{2}\cos 2x\,dx = \left[-\frac{x}{2}\cos 2x\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2x\,dx$

$= \left[\frac{-x}{2}\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\pi/2}$

$= \left(-\frac{\pi}{4}(-1) + 0\right) - \left(0 + 0\right) = \pi/4$

2. $I = 16 - 16(0.5)^t = 16 - 16\left(e^{\ln 0.5}\right)^t = 16 - 16\,e^{(\ln 0.5)t}$

$\therefore \frac{dI}{dt} = -16\ln(0.5)e^{(\ln 0.5)t} = -16\ln(0.5)(0.5)^t$

At $t = 3$, $\frac{dI}{dt} = -16\ln(0.5)\left(\frac{1}{8}\right) = -2\ln 0.5$

$= \ln(0.5)^{-2}$

$= \ln 4$

3(a) Let $\frac{5}{(x-1)(3x+2)} = \frac{A}{x-1} + \frac{B}{3x+2}$

$\times(x-1)(3x+2)$, $5 = A(3x+2) + B(x-1)$

Let $x = 1 \Rightarrow 5A = 5,\ A = 1$

Let $x = -2/3$, $-5/3\,B = 5$, $B = 5(-3/5) = -3$

$\therefore \frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$

b) $\int\frac{5}{(x-1)(3x+2)}\,dx = \int\frac{1}{x-1} - \frac{3}{3x+2}\,dx = \ln(x-1) - \ln(3x+2) + c$

c) $(x-1)(3x+2)\,dy/dx = 5y$, separate the variables:

$\frac{1}{y}\,dy/dx = \frac{5}{(x-1)(3x+2)}$

$\int\frac{1}{y}\,dy/dx\,dx = \int\frac{1}{y}\,dy = \ln y = \int\frac{5}{(x-1)(3x+2)}\,dx = \ln(x-1) - \ln(3x+2) + c$

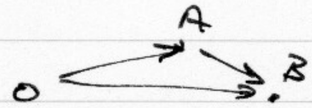
$\therefore \ln y = \ln\left(\frac{x-1}{3x+2}\right) + c$, $y = e^c\left(\frac{x-1}{3x+2}\right)$.

$e^c = \text{constant} = A, \quad y = A\left(\frac{x-1}{3x+2}\right)$

At $x = 2$,

$$y = A\left(\frac{2-1}{6+2}\right) = \frac{A}{8} = 8 \quad \therefore A = 64$$

$$y = 64\left(\frac{x-1}{3x+2}\right)$$



4. (a) $\vec{OA} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

$\vec{OA} + \vec{AB} = \vec{OB} \quad \therefore \quad \vec{AB} = \vec{OB} - \vec{OA}$

$$\vec{AB} = \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$$

(b) General position vector on l_1:

$$\underline{r} = \vec{OA} + \lambda \vec{AB}$$

$$= \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$$

(c) $\vec{AC} = \begin{pmatrix} 2 \\ p \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ p+3 \\ -6 \end{pmatrix}$.

Need $\vec{AC} \cdot \vec{AB} = 0$ if $\perp$ to line l_1

$$\therefore \begin{pmatrix} 1 \\ p+3 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix} = -3 + 5p + 15 + 18 = 0$$

$$5p + 30 = 0$$

$$p = -6$$

(d) with $p = -6$, $\vec{AC} = \begin{pmatrix} 1 \\ -3 \\ -6 \end{pmatrix}$

$$|AC| = \sqrt{1^2 + 3^2 + 6^2} = \sqrt{46}$$

5. $(2-3x)^{-2} = \left[2\left(1 - \tfrac{3}{2}x\right)\right]^{-2} = \frac{1}{4}\left(1 - \tfrac{3}{2}x\right)^{-2}$

$$(1+X)^n = 1 + nX + \frac{n(n-1)}{2}X^2 + \frac{n(n-1)(n-2)}{3!}X^3 + \dots$$

$$\therefore \left(1-\tfrac{3}{2}x\right)^{-2} = 1 - 2\left(-\tfrac{3}{2}x\right) + \frac{(-2)(-3)}{2}\left(\tfrac{9}{4}x^2\right) + \frac{(-2)(-3)(-4)}{6}\left(\tfrac{-27}{8}x^3\right) + \dots$$

$$= 1 + 3x + \frac{27}{4}x^2 + \frac{27}{2}x^3 + \dots$$

$\therefore \frac{1}{4}(1-\tfrac{3}{2}x)^{-2} = \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{27}{8}x^3 + \ldots$

5 (b) $f(x) = \dfrac{a+bx}{(2-3x)^2} = \frac{1}{4}a + \frac{3}{4}ax + \frac{27}{16}ax^2 + \frac{27}{8}ax^3$
$+ \frac{1}{4}bx + \frac{3}{4}bx^2 + \frac{27}{16}bx^3$

$\therefore \frac{3}{4}a + \frac{1}{4}b = 0$ (x-coefficients), $3a+b=0$

$\frac{27}{16}a + \frac{3}{4}b = \frac{9}{16}$ (x^2 coefficients)

($\times 16$): $27a + 12b = 9$, $9a + 4b = 3$

$3a+b=0 \;\therefore$ $\quad \underline{9a+3b = 3\times 0 = 0}$

subtract: $b = 3$, $a = -1$

(c) $\frac{27}{8}a + \frac{27}{16}b = 27\left(\frac{-1}{8} + \frac{3}{16}\right) = \frac{27}{16}$ = coeff. of x^3

6. $x = \ln t$, $y = t^2 - 2$, $t > 0$

(a) $dx/dt = \frac{1}{t}$, $dy/dt = 2t$, $dy/dx = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{(1/t)} = 2t^2$

At $t=3$, $dy/dx = 2\times 3^2 = 18$

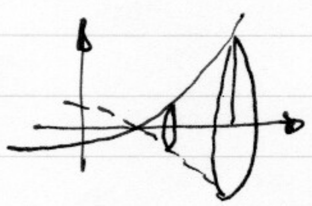
$x = \ln 3$, $y = 3^2 - 2 = 7$

gradient of normal $= \dfrac{-1}{18}$

$y - 7 = -\frac{1}{18}(x - \ln 3)$

$y = -\frac{1}{18}x + \frac{\ln 3}{18} + 7$

(b) $x = \ln t \;\therefore\; t = e^x$ $\qquad y = t^2 - 2 = e^{2x} - 2$

(c) 

$V = \int_a^b \pi y^2\,dx = \pi \int_{\ln 2}^{\ln 4} (e^{2x} - 2)^2\,dx$

$\frac{V}{\pi} = \int_{\ln 2}^{\ln 4} e^{4x} - 4e^{2x} + 4\,dx = \left[\frac{e^{4x}}{4} - 2e^{2x} + 4x\right]_{\ln 2}^{\ln 4}$

$= \left(\frac{4^4}{4} - 2\times 4^2 + 4\ln 4\right) - \left(\frac{2^4}{4} - 2\times 2^2 + 4\ln 2\right) = (64 - 32 + \ln 256) - (4 - 8 + \ln 16)$

$= 36 + \ln\left(\frac{256}{16}\right) = 36 + \ln 16 \quad \therefore V = \pi(36 + \ln 16)$

7. $I = \int_2^5 \frac{1}{4+\sqrt{x-1}}\,dx$

a)

x	2	3	4	5
y	0.2	0.1847	0.1745	0.1667

nb $\sqrt{x-1}$ is the positive square root (otherwise you don't get 0.2 etc).

b) $h = \frac{5-2}{3} = 1$ or just $3-2=1$

$I \approx h/2\,(y_0+y_3+2(y_1+y_2)) = \frac{1}{2}(0.2+0.1667+2(0.1847+0.1745))$

$= 0.54255$

$= 0.543$ to 3 dp.

c) Let $x = (u-4)^2+1$, $dx/du = 2(u-4) = 2u-8$

then $(u-4)^2 = x-1$

$u = 4+\sqrt{x-1}$

At $x=5$, $u = 4+\sqrt{4} = 6$

nb since we chose the positive root in (a), we use $u = 4+\sqrt{4}$ here, not $u = 4-\sqrt{4}$!!

At $x=2$, $u = 4+\sqrt{1} = 5$

$$I = \int_{x=2}^{5} \frac{1}{4+\sqrt{x-1}}\,dx = \int_{u=5}^{6} \frac{1}{u}\frac{dx}{du}\,du = \int_5^6 \frac{2u-8}{u}\,du$$

$$= \int_5^6 2-8u^{-1}\,du = \left[2u-8\ln u\right]_5^6$$

$$= (12-8\ln 6)-(10-8\ln 5) = 2+8\ln(5/6)$$

[check, $I = 0.54143\ldots$, close to (b) answer].