C4 Jan 06

1. $3x^2 + 4y^2 - 2x + 6xy - 5 = 0$

$6x + 8y\,\frac{dy}{dx} - 2 + 6y + 6x\frac{dy}{dx} = 0$

$\therefore (8y + 6x)\frac{dy}{dx} = 2 - 6x - 6y.$

$\frac{dy}{dx} = \frac{2 - 6x - 6y}{6x + 8y} = \frac{1 - 3x - 3y}{3x + 4y}.$

At $(1, -2)$, $\frac{dy}{dx} = \frac{1 - 3 + 6}{3 - 8} = \frac{4}{-5} = -4/5$

Tangent is $y + 2 = -4/5(x - 1)$, $5y + 10 = -4x + 4$,
$4x + 5y + 6 = 0$.

2(a)

x	0	$\pi/16$	$\pi/8$	$\frac{3\pi}{16}$	$\pi/4$
y	1	1.01959	1.08239	1.20269	1.41421

(b) $h = \pi/16$

$\int_0^{\pi/4} \sec x\,dx \approx h/2\left(y_0 + y_4 + 2(y_1 + y_2 + y_3)\right)$
$= 0.885855$
$= 0.8859$ to 4 d.p.

(c) ANS: $\ln(1 + \sqrt{2}) = 1.0051$, error 0.51%

3. $u^2 = 2x - 1 \quad \therefore 2x = u^2 + 1, \quad x = \frac{1}{2}(u^2 + 1).$

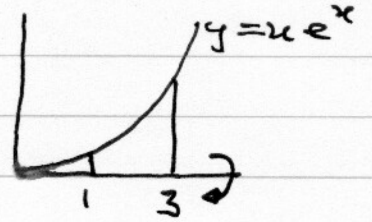
$dx/du = \frac{1}{2}(2u) = u. \quad x = 1 \to u = 1$
$x = 5 \to u = 3$

$\int_1^5 \frac{3x}{\sqrt{2x - 1}}\,dx = \int_{x=1}^{5} \frac{3/2(u^2 + 1)}{u}\,\frac{dx}{du}\,du$

$= \int_{u=1}^{3} 3/2(u^2 + 1)\cancel{\frac{u}{du}}\,du = \left[\frac{3}{2}\frac{u^3}{3} + \frac{3}{2}u\right] = \left[\frac{u^3}{2} + 3/2u\right]_1^3$

$= \left(\frac{27}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + 3/2\right) = 18 - 2 = 16$

4.



$V = \int_1^3 \pi y^2\,dx = \pi\int_1^3 x^2 e^{2x}\,dx$

$\int uw\,dx = uW - \int u'W\,dx, \quad u = x^2, \; w = e^{2x}$

$$\therefore V = \left[\pi x^2 \frac{e^{2x}}{2}\right]_1^3 - \pi\int (2x)\frac{e^{2x}}{2}\,dx = \left[\pi \frac{x^2 e^{2x}}{2}\right] - \pi\int_1^3 x e^{2x}\,dx$$

$$\int x e^{2x}\,dx = x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$

$$= x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} = \frac{e^{2x}}{4}(2x-1) + c$$

$$\therefore V = \left[\pi \frac{e^{2x}}{4}\left(2x^2 - 2x + 1\right)\right]_1^3 = \frac{\pi}{4}\left[13e^6 - e^2\right] = \frac{\pi e^2}{4}\left(13e^4 - 1\right)$$

5(a) $f(x) = \dfrac{3x^2+16}{(1-3x)(2+x)^2} = \dfrac{A}{1-3x} + \dfrac{B}{2+x} + \dfrac{C}{(2+x)^2}$

$\times\ (1-3x)(2+x)^2$ each side:

$3x^2+16 = A(2+x)^2 + B(1-3x)(2+x) + C(1-3x)$

Let $x = 1/3$, $16\tfrac{1}{3} = \left(\tfrac{7}{3}\right)^2 A$, $\tfrac{49}{3} = \tfrac{49}{9}A$, $A = 3$

Let $x = -2$, $28 = 7C$, $C = 4$

Let $x = 0$, $16 = 4A + 2B + C = 12 + 2B + 4 = 16 + 2B$ $\therefore B = 0$

(b) $(1-3x)^{-1} = 1 + (-1)(-3x) + \dfrac{(-1)(-2)}{2!}(-3x)^2 + \dfrac{(-1)(-2)(-3)}{3!}(-3x)^3 + \dots$

$= 1 + 3x + 9x^2 + 27x^3 + \dots$

$\left(1 + x/2\right)^{-2} = 1 + (-2)\left(\dfrac{x}{2}\right) + \dfrac{(-2)(-3)}{2}\left(\dfrac{x}{2}\right)^2 + \dfrac{(-2)(-3)(-4)}{3!}\left(\dfrac{x}{2}\right)^3 + \dots$

$= 1 - x + \dfrac{3}{4}x^2 - \dfrac{x^3}{2} + \dots$

$\therefore 3(1-3x)^{-1} + 4(2+x)^{-2} = 3(1-3x)^{-1} + 4\left[2\left(1 + x/2\right)\right]^{-2}$

$= 3(1-3x)^{-1} + \left(1 + x/2\right)^{-2} = 3 + 9x + 27x^2 + 81x^3 + \dots$

$+ 1 - x + \tfrac{3}{4}x^2 - \dfrac{x^3}{2} + \dots$

$= 4 + 8x + 27\tfrac{3}{4}x^2 + 80\tfrac{1}{2}x^3 + \dots$

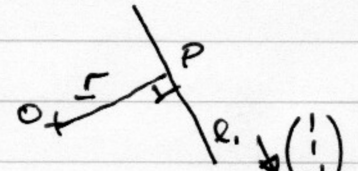
6. (a) $\underline{r} = \begin{pmatrix} 8 \\ 12 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} 4 \\ 8 \\ a \end{pmatrix}$ $\quad 8+\lambda = 4 \therefore \lambda = -4$, check $12+\lambda = 12-4 = 8 \checkmark$

$\therefore a = 14 - \lambda = 14 + 4 = 18$

$\underline{b} = \begin{pmatrix} b \\ 13 \\ 13 \end{pmatrix}$ $\quad 12 + \lambda = 13 \therefore \lambda = 1$, check $14 - \lambda = 13 \checkmark$

$\therefore b = 8 + \lambda = 9$

b) 

Need $\underline{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$ so OP $\perp$ to l_1

$$\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 8+\lambda + 12+\lambda - 14+\lambda = 0$$

$\therefore 6 + 3\lambda = 0, \ \lambda = -2$

$\vec{OP} = \begin{pmatrix} 8-2 \\ 12-2 \\ 14+2 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 16 \end{pmatrix} \therefore$ P is $(6, 10, 16)$.

c) $|\vec{OP}| = \sqrt{6^2 + 10^2 + 16^2} = \sqrt{392} = \sqrt{8 \times 49} = 14\sqrt{2}$

7 (a) $V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$ (= surface area $\checkmark$).

(b) $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \quad \therefore dr/dt = \frac{dV/dt}{dV/dr} = \frac{\left(\frac{1000}{(2t+1)^2}\right)}{4\pi r^2}$

$\therefore \frac{dr}{dt} = \frac{250}{\pi r^2 (2t+1)^2}$

(c) $\frac{dV}{dt} = 1000(2t+1)^{-2} \quad \therefore \int \frac{dV}{dt} dt = \int dV = \int 1000(2t+1)^{-2} dt$

$V = 1000 \frac{(2t+1)^{-1}}{-2} + c = -500(2t+1)^{-1} + c$

At $t = 0$, $V = -500 + c = 0 \therefore c = 500$

$V = 500\left(1 - \frac{1}{2t+1}\right) = 500\left(\frac{2t}{2t+1}\right) = \frac{1000t}{2t+1}$ cm^3

7 (a)(i) $t=5, \quad V = \frac{5000}{11} = \frac{4}{3}\pi r^3 \quad \therefore r = 4.7698$

$= 4.77 \text{ cm}$

(ii) $\frac{dr}{dt} = \frac{250}{\pi r^2 (2t+1)^2} = \cancel{\dots}\, 0.0289$

$\approx 0.029 \text{ cm/s}, \quad 2.9 \times 10^{-2} \text{ cm/s}.$

8. 

$x = t - 2\sin t, \quad y = 1 - 2\cos t$

$\therefore dx/dt = 1 - 2\cos t = y.$

a) At $y=0$, $1 - 2\cos t = 0 \quad \therefore \cos t = 1/2$,

$t = \cos^{-1}(1/2) = \pm \pi/3 + n360. \qquad 0 \le t < 2\pi$

$\therefore t = \pi/3, \quad 5\pi/3.$

(b) $\text{Area} = \int y\,dx = \int_{t=\pi/3}^{5\pi/3} (1-2\cos t)\, \frac{dx}{dt}\, dt = \int_{\pi/3}^{5\pi/3} (1-2\cos t)^2\, dt$

(c) $(1-2\cos t)^2 = 1 - 4\cos t + 4\cos^2 t.$

$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - (1-\cos^2 t) = 2\cos^2 t - 1$

$\therefore 2\cos^2 t = 1 + \cos 2t$

$\text{Area} = \int_{\pi/3}^{5\pi/3} 1 - 4\cos t + 2(1+\cos 2t)\, dt = \int_{\pi/3}^{5\pi/3} 3 - 4\cos t + 2\cos 2t\, dt$

$= \left[3t - 4\sin t + \sin 2t\right]_{\pi/3}^{5\pi/3}$

$= \left(5\pi - 4\left(\frac{-\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right) - \left(\pi - 4\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right)$

$= 4\pi + 3\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right) = 4\pi + 3\sqrt{3}$