C3 June 2010

1(a) $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$

$$\therefore \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

(b) $$\frac{2\sin 2\theta}{1+\cos 2\theta} = 2\tan\theta = 1 \qquad \therefore \tan\theta = \tfrac{1}{2}$$

$$\theta = \tan^{-1}(\tfrac{1}{2}) + n180° = 26.57° + n180$$

$$\therefore \theta = -153.4°,\ 26.6°$$

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2. $$y = \frac{3}{(5-3x)^2} = 3(5-3x)^{-2}$$

$$dy/dx = 3(-2)(5-3x)^{-3}(-3) = 18(5-3x)^{-3}$$

At $x=2$, $dy/dx = m_T = 18(5-6)^{-3} = -18$

$\therefore m_N = \frac{-1}{-18} = \frac{1}{18}$. On C at $x=2$, $y = \frac{3}{(5-6)^2} = 3$.

$$y - 3 = \tfrac{1}{18}(x-2)$$

$$18y - 54 = x - 2, \qquad x - 18y + 52 = 0$$

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3. $f(x) = 4\operatorname{cosec} x - 4x + 1$ $\qquad$ Radians

(a) $f(1.2) = 0.4917$, $f(1.3) = -0.0487$ } Sign change in $f(x)$ and $f(x)$ is continuous [both $1.2, 1.3 < \pi/2$] so there is a root in this interval.

(b) $f(x) = 0$

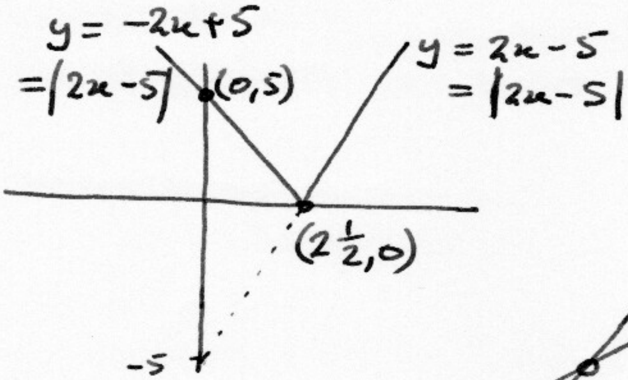
$$\therefore \frac{4}{\sin x} - 4x + 1 = 0, \quad \frac{1}{\sin x} - x + \frac{1}{4} = 0,$$

$$x = \frac{1}{\sin x} + \frac{1}{4}$$

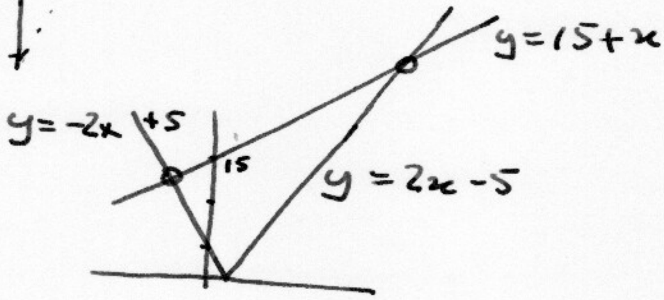
(c) $x_1 = 1.3038$, $x_2 = 1.2867$, $x_3 = 1.2917$

(d) Sign change, $f(1.2905) = 0.00045$, $f(1.2915) = -0.0048$ } so there's a root between these, it rounds to 1.291 to 3 d.p.

4. (a) $f(x) \equiv |2x-5|$



(b)



$15+x = -2x+5$

$3x = -10$

$\boxed{x = -\frac{10}{3}}$

$2x-5 = 15+x$

$\boxed{x = 20}$

(c) $g(2) = 4-8+1 = -3$

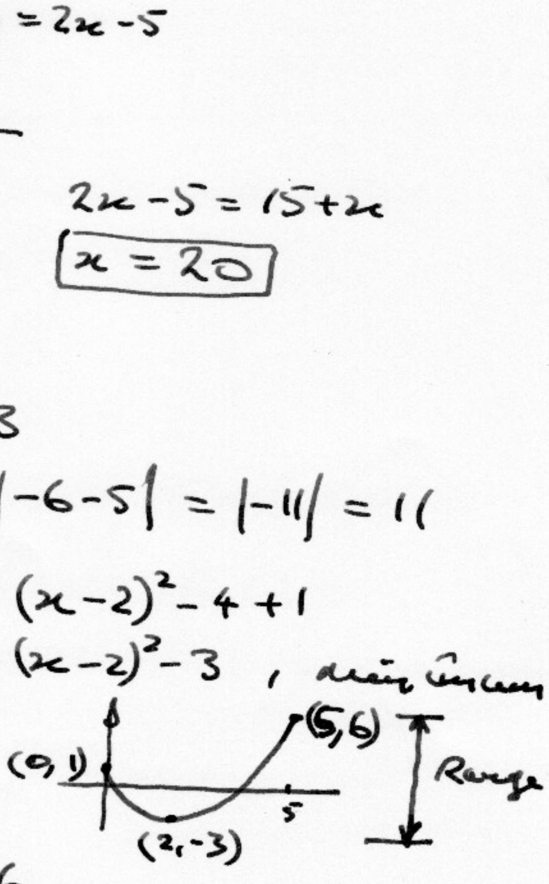
$f(g(2)) = f(-3) = |-6-5| = |-11| = 11$

(d) $g(x) = x^2-4x+1 = (x-2)^2-4+1$

$= (x-2)^2-3$, minimum $g(2) = -3$

$x=0 \Rightarrow g(0) = 1$

$x=5 \Rightarrow g(5) = 6$



Range $-3 \le g(x) \le 6$

5. (a) $y = (2x^2-5x+2)e^{-x}$

At $x=0$, $y = 2e^0 = 2$, coordinates $(0,2)$.

(b) $e^{-x} \neq 0$ for any real x, $\therefore$ cuts x-axis only where

$2x^2-5x+2 = 0$ $\qquad ac = 4 = -1 \times -4$

$(2x - \frac{1}{1})(1x - \frac{4}{2}) = (2x-1)(x-2) = 0$, $\quad x = 1/2$ or 2

(c) $dy/dx = (4x-5)e^{-x} + (2x^2-5x+2)(-e^{-x}) = (-2x^2+9x-7)e^{-x}$

(d) Turning points at $dy/dx = 0$ $\therefore$ $2x^2-9x+7=0$

$ac = 14 = -2 \times -7$, $\quad (2x-\frac{7}{1})(1x-\frac{2}{2}) = (2x-7)(x-1) = 0$

5(d) $x-1=0 \to x=1,\ y=(2-5+2)e^{-1}=-e^{-1}$

$2x-7=0,\ x=7/2,\ y=\left(2\left(\frac{49}{4}\right)-\frac{35}{2}+2\right)e^{-7/2}=9e^{-7/2}$

$\therefore$ Turning points are $(1,-e^{-1})$ and $\left(\frac{7}{2}, 9e^{-7/2}\right)$

Check - first has negative y, second positive y ✓.

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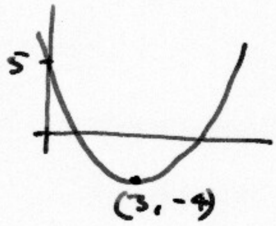
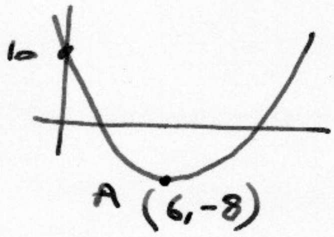
6. 

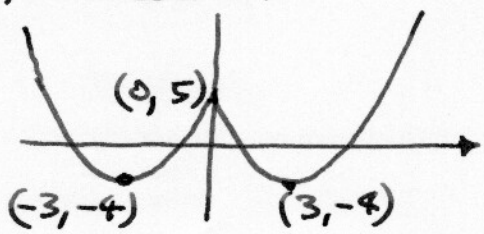
(a) (i) $y=|f(x)|$, A becomes $(3,4)$

(ii) $y=2f(\frac{1}{2}x)$, stretched $\times 2$ in both x and y directions



Sketches not required

(b) $y=f(|x|)$, reflects in y-axis:



(c) $y=x^2$ has translated $\xrightarrow{3}$, $\downarrow 4$.

$f(x)=(x-3)^2-4$

$=x^2-6x+9-4=x^2-6x+5$

Let $g(x)=x^2$, $f(x)=g(x-3)-4$

(d) f is a "many : one" (or two : one) mapping, the inverse would not give a unique x value for each y value.

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7. (a) $R\sin(\theta-\alpha)=R\sin\theta\cos\alpha-R\cos\theta\sin\alpha=2\sin\theta-1.5\cos\theta$

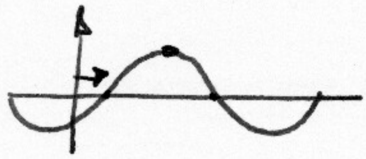
$\therefore R\cos\alpha=2,\ R\sin\alpha=1.5,\quad R=\sqrt{2^2+1.5^2}=\sqrt{6.25}=2.5$

$\tan\alpha=\frac{1.5}{2}\quad \therefore \alpha=0.6435$ radians

$2\sin\theta-1.5\cos\theta=2.5\sin(\theta-0.6435)$.

7 (b)(i) Max value of $2.5\sin(\theta - 0.6435)$ is 2.5.

(ii) Sine curve translated $\rightarrow 0.6435$



$\therefore$ Maximum at $\theta - 0.6435 = \pi/2$,

$\theta = 2.2143$

(c) Maximum value $H = 6 + 2.5 = 8.5\,m$, occurs when

$\theta = \dfrac{4\pi t}{25} = 2.2143 \quad \therefore t = \dfrac{25 \times 2.2143}{4\pi} = 4.4052$

$= 4.41$ to 2d.p.

(d) At $H = 7$, $2.5\sin(\theta - 0.6435) = 7 - 6 = 1$

$\theta - 0.6435 = \sin^{-1}\left(\frac{1}{2.5}\right) = 0.4115, \quad \pi - 0.4115 + n2\pi$

$0 \le t < 12$ implies $0 \le \theta \le 1.92\pi$

$\theta = 0.4115 + 0.6435 = 1.055, \quad t = 2.0989 = 2h\ 6\text{mins}$

$\theta = \pi - 0.4115 + 0.6435 = 3.3736, \quad t = 6.7115 = 6h\ 43\text{mins}$

$\therefore H = 7$ at 14:06 and at 18:43

[Also 12.5 hours before each of these, since $\frac{4\pi(12.5)}{25} = 2\pi$, but $t \ge 0$ implies times before midday not required].

8.(a) $ac = -10 = 10 \times -1$, $2x^2 + 9x - 5 = \left(2x - \frac{1}{1}\right)\left(1x + \frac{10}{2}\right)$

$\therefore \dfrac{2x^2 + 9x - 5}{x^2 + 2x - 15} = \dfrac{(2x-1)(x+5)}{(x-3)(x+5)} = \dfrac{2x-1}{x-3}$

(b) $\ln(2x^2 + 9x - 5) - \ln(x^2 + 2x - 15) = \ln\left(\dfrac{2x^2+9x-5}{x^2+2x-15}\right) = 1$

$\therefore \ln\left(\dfrac{2x-1}{x-3}\right) = 1, \quad \dfrac{2x-1}{x-3} = e^1$

$2x - 1 = ex - 3e, \quad ex - 2x = -1 + 3e$

$= (e-2)x$

$\therefore x = \dfrac{3e-1}{e-2}$