June '07 C3

1 (a) $\ln x + \ln 3 = \ln 6$

$= \ln(3x)$

$e^{\ln(3x)} = e^{\ln 6}$, $3x = 6$, $x = 2$.

(b) $e^x + 3e^{-x} = 4$.

$(\times e^x)$: $e^{2x} + 3 = 4e^x$

$e^{2x} \cancel{4} - 4e^x + 3 = 0$.

Let $y = e^x$, $y^2 - 4y + 3 = (y-1)(y-3) = 0$

$\Rightarrow y = 1$ or 3 $= e^x$

$\therefore x = \ln 1 = \underline{0}$

or $\underline{\ln 3}$

---

2. $f(x) = \dfrac{2x+3}{x+2} - \dfrac{9+2x}{2x^2+3x-2}$

Factorise $2x^2+3x-2$

$2x - 2 = -4 = 4x - 1$

$2x^2+3x-2 = 2x^2+4x-x-2 = 2x(2x+2)-1(x+2)$

$= (2x-1)(x+2)$.

$\therefore f(x) = \dfrac{(2x+3)(2x-1) - 9 - 2x}{(x+2)(2x-1)} = \dfrac{4x^2+6x-2x-3-9-2x}{(\ )(\ )}$

$= \dfrac{4x^2+2x-12}{(\ )(\ )} = \dfrac{2(2x^2+x-6)}{(x+2)(2x-1)}$.

(c) $\frac{d^2y}{dx^2} = (2x+2)e^x + (x^2+2x)e^x$

$= (x^2+4x+2)e^x$

(d) At $x=0$, $d^2y/dx^2 = 2e^0 = 2$

(dy/dx increases with x, hence a minimum)

At $x=-2$, $d^2y/dx^2 = (4-8+2)e^{-2} = -2e^{-2}$

(dy/dx decreases with x, hence a maximum).

4. $f(x) = -x^3+3x^2-1$

(a) $f(x) = 0$

Looking at the answer, we want to write as

$x^2 = \frac{1}{3-x}$,

$(f(x)+1)$

$= 3x^2-x^3 = 1$

$= (3-x)x^2$

$\therefore x^2 = \frac{1}{3-x}$, $x = \sqrt{\frac{1}{3-x}}$

(b) $0.6 =$ (x_0)

$\sqrt{(1\div(3\text{-ANS}))} = 0.6455$ (x_2)

" $= 0.6517$ (x_3)

" 0.6526 (x_4)

(c) "$f(0.6525)$".

$0.6525=$, $0-\text{ANS}^3+3\times\text{ANS}^2-1 = -5.4\times10^{4}$

$0.6535=$, ... $= 2.1\times10^{-3}$

The sign of $f(x)$ changes and $f(x)$ is continuous so there is a root in this interval and it rounds to 0.653 to 3 d.p.

5. $f(x) = \ln(2x-1)$

$g(x) = \dfrac{2}{x-3}$

(a) $f(g(4)) = f\left(\dfrac{2}{4-3}\right) = f(2) = \ln(2\times2-1)$

$= \ln 3$

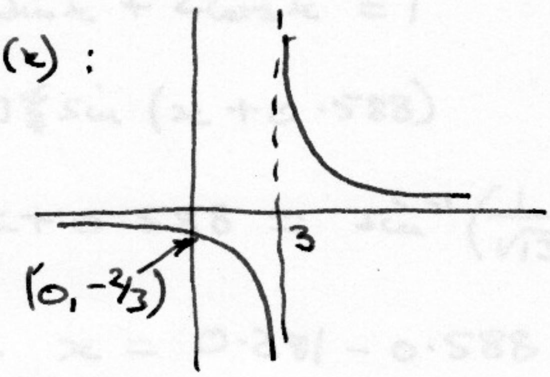
(b) $y = f(x) = \ln(2x-1)$.

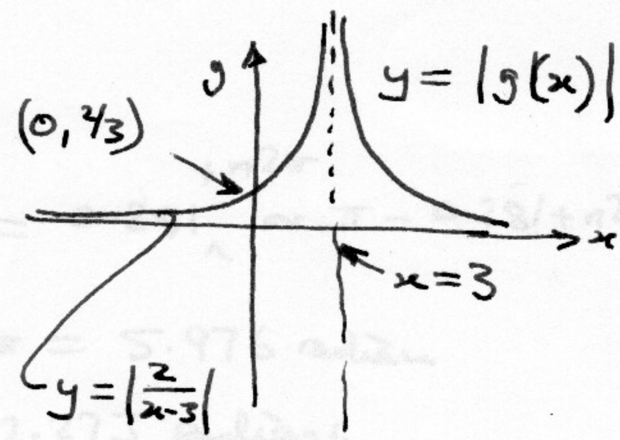
Swap x & y $\Rightarrow$ $x = \ln(2y-1)$

Make y the subject, $e^x = 2y-1$,

$2y = e^x + 1$, $y = \frac{1}{2}(e^x+1) = f^{-1}(x)$, $x \in R$, domain

(c) $g(x)$:



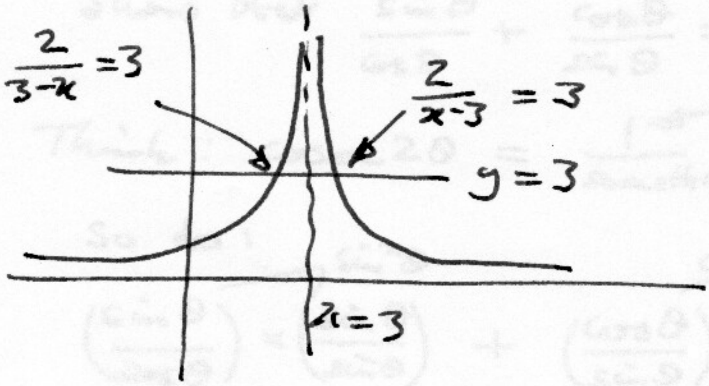


$y = \left|\dfrac{2}{x-3}\right|$

$= \dfrac{2}{3-x}$ here, ~~xxx~~

$x = 0$, $y = 2/3$

(d)



For $\dfrac{2}{x-3} = 3$, $2 = 3x - 9$, $3x = 11$, $\underline{x = 4/3}$

For $\dfrac{2}{3-x} = 3$, $2 = 9 - 3x$, $-7 = -3x$, $\underline{x = 7/3}$

6. (a) $3\sin x + 2\cos x = R\sin(\alpha + x)$

$= R\cos\alpha \sin x + R\sin\alpha \cos x$.

$\therefore$ Need $R\cos\alpha = 3$, $R\sin\alpha = 2$.

$R^2 = 3^2 + 2^2 = 9 + 4 = 13$, $R = \sqrt{13}$

$\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha} = 2/3$, $\alpha = 0.588$ radian

$\therefore 3\sin x + 2\cos x = \sqrt{13}\sin(x + 0.588)$.

(b) $(\sqrt{13})^4 = 13^2 = 169$

(c) $3\sin x + 2\cos x = 1$

$= \sqrt{13}\sin(x + 0.588)$

$x + 0.588 = \sin^{-1}\left(\dfrac{1}{\sqrt{13}}\right) = 0.281 + n2\pi$ or $\pi - 0.281 + n2\pi$.

$\therefore x = 0.281 - 0.588 + 2\pi = 5.976$ radian

or $\pi - 0.281 - 0.588 = 2.273$ radians.

7 (a) "Show that $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = 2\operatorname{cosec} 2\theta$"

Think : $\operatorname{cosec} 2\theta = \dfrac{1}{\text{something}}$, $\sin^2\theta + \cos^2\theta = 1$

So do :

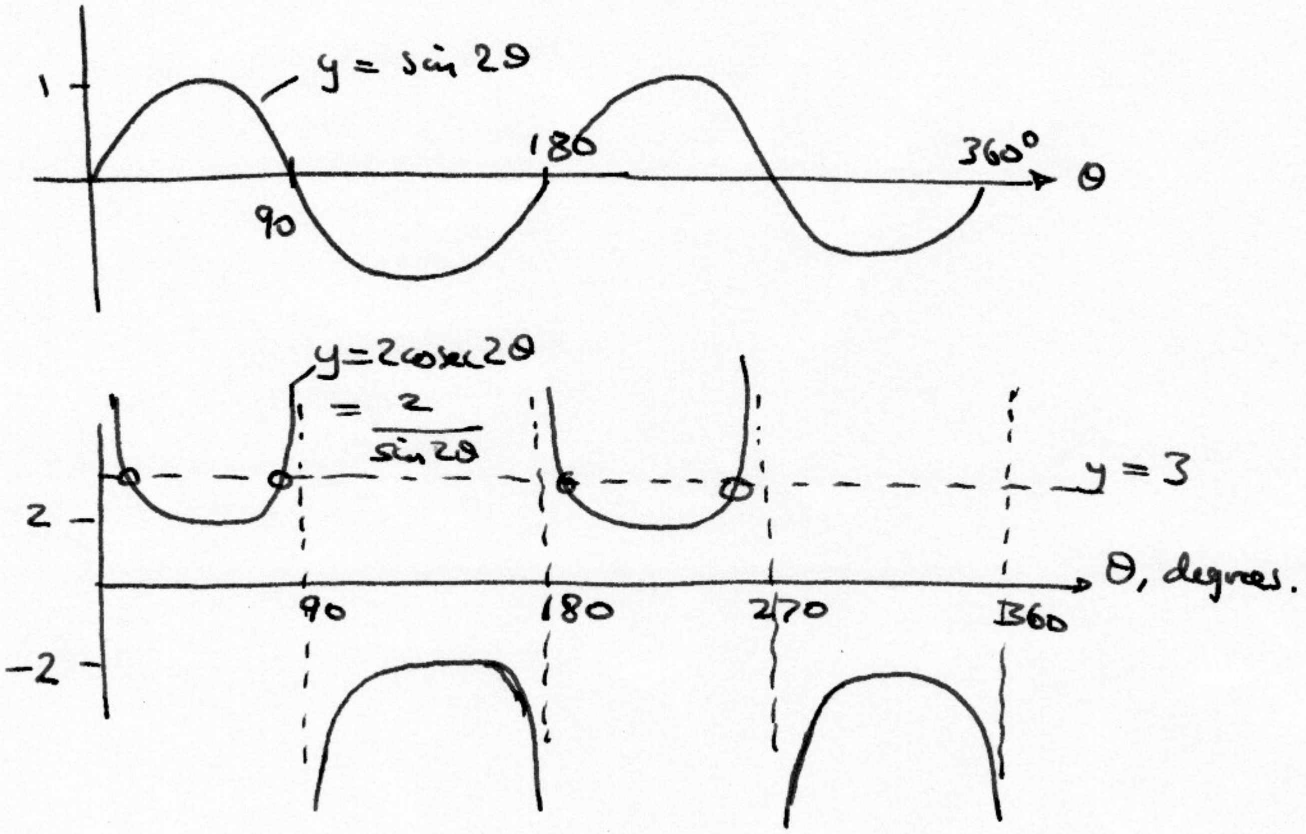
$$\underbrace{\left(\frac{\sin\theta}{\cos\theta}\right) \times \left(\frac{\sin\theta}{\sin\theta}\right)}_{\sin^2\theta} + \underbrace{\left(\frac{\cos\theta}{\sin\theta}\right) \times \left(\frac{\cos\theta}{\cos\theta}\right)}_{\cos^2\theta}$$

just adding fractions, using a common denominator

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} = \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = \underline{\underline{2\operatorname{cosec} 2\theta}}$$

7(b)



(c) $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 3$, solutions are intersection points of $y = 3$ and $y = 2\operatorname{cosec} 2\theta$.

$2\operatorname{cosec} 2\theta = 3$, $\operatorname{cosec} 2\theta = \frac{1}{\sin 2\theta} = \frac{3}{2}$,

$\sin 2\theta = 2/3$,

$2\theta = 41.81 + n360$ or $(180 - 41.81) + n360$

$\theta = 20.9052 + n180$ or $69.0948 + n180$

$\therefore$ roots are $20.9°$, $69.1°$, $200.9°$, $249.1°$

8. $x = De^{-1/8 t}$

(a) $t=5$, $x = 10e^{-5/8} = 5.353\,\text{mg}$

(b) Assuming that the second dose doesn't affect the decay rate of the first,

x_1 (6 hours after first dose) $= 10e^{-6/8}$

x_2 (1 hour after second dose) $= 10e^{-1/8}$

Total remaining in bloodstream $x_1 + x_2$

$= 10\left(e^{-6/8} + e^{-1/8}\right) = 13.549\,\text{mg}.$

or: After 5 hours, amount in blood $= 5.353$ (from (a)) $+ 10$ $= 15.353$.
Then after a further hour,
$x = 15.353e^{-1/8}$
$= 13.549\,\text{mg}.$

(c) T hours after second dose,

$t = T + 5.$

$$3 = 10e^{-1/8 t} + 10e^{-1/8 T}$$

$$\frac{3}{10} = e^{-1/8(T+5)} + e^{-1/8 T}$$

$$= e^{-1/8 T}\left(e^{-5/8} + 1\right)$$

$$e^{-T/8} = \frac{0.3}{1 + e^{-5/8}} = 0.1954$$

$$-T/8 = \ln(0.1954) = -1.633$$

$T = 13.06$ hours (to 4 s.f.).

or:
$$3 = 15.353e^{-1/8 T}$$

$$\frac{3}{15.353} = e^{-1/8 T}$$

$$\ln\left(\frac{3}{15.353}\right) = \frac{-T}{8}$$

$T = 13.06$ hours