JUNE 2006 C3

1 (a) Simplify $\dfrac{3x^2-x-2}{x^2-1}$.

Think: this cannot be simplified unless $3x^2-x-2$ and x^2-1 have a common factor, so try factorising it

$3\times -2 = -6$, factors of -6 that add to -1 are $\underline{-3\times 2}$.

$$3x^2-3x+2x-2 = 3x(x-1)+2(x-1)$$
$$= (3x+2)(x-1)$$

and $x^2-1 = (x+1)(x-1)$

so it simplifies to $\dfrac{3x+2}{x+1} = \underbrace{\dfrac{3x+3-1}{x+1} = 3-\dfrac{1}{x+1}}_{\text{optional.}}$

(b) $\dfrac{3x^2-x-2}{x^2-1} = \dfrac{3x^2+2x}{x(x+1)}$

$$\frac{3x^2-x-2}{x^2-1} - \frac{1}{x(x+1)} = \frac{3x^2+2x-1}{x(x+1)}$$

Factors of -3 that add to 2 are 3×-1.

$$3x^2+2x-1 = 3x^2+3x-x-1 = 3x(x+1)-1(x+1)$$
$$= (3x-1)(x+1)$$

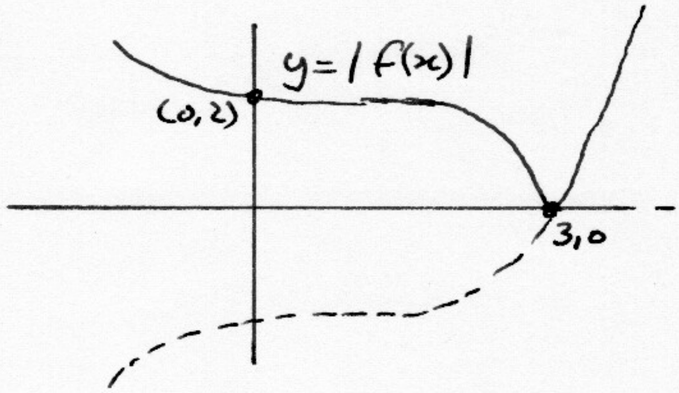
$\therefore$ Answer is $\dfrac{3x-1}{x}$

2. (a) $\frac{d}{dx}\left(e^{3x} + \ln 2x\right) = 3e^{3x} + 2\left(\frac{1}{2x}\right) = 3e^{3x} + \frac{1}{x}$

(b) $\frac{d}{dx}\left(5+x^2\right)^{3/2} = 3/2\left(5+x^2\right)^{1/2}(2x)$

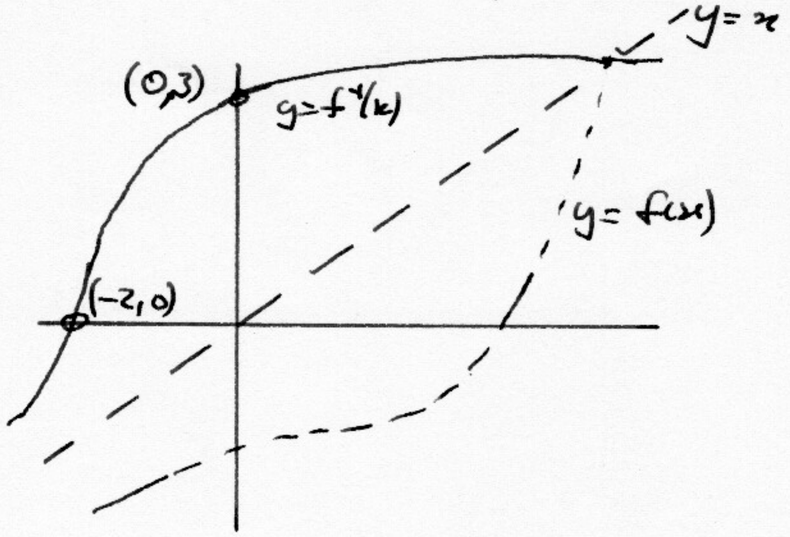
$= 3x\left(5+x^2\right)^{1/2}$

3. (a)

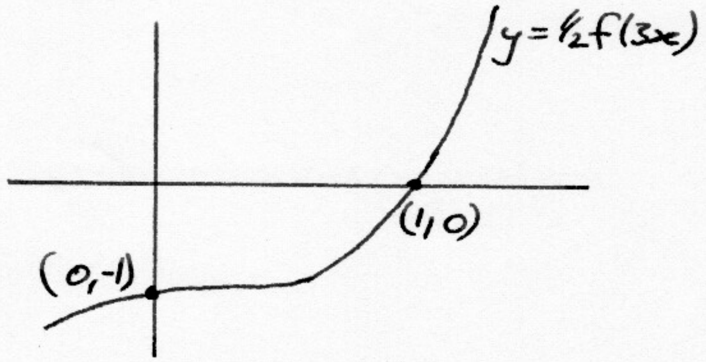


Think: "y always positive"

(b) $y = f^{-1}(x)$, reflect in $y = x$



(c)



(stretch by factor 1/3 in x direction, ½ in y direction).

4. $T = 400e^{-0.05t} + 25$

(a) At $t = 0$, $T = 400e^{0} + 25 = 425\,°C$

(b) At $T = 300$,

$400e^{-0.05t} + 25 = 300$

$e^{-0.05t} = \frac{275}{400}$, $t = 7.494$ minutes (calculator).

$= 7.49$ minutes to 3 s.f.

(c) "Rate" $= \frac{dT}{dt} = 400 \times (-0.05)\,e^{-0.05t}$

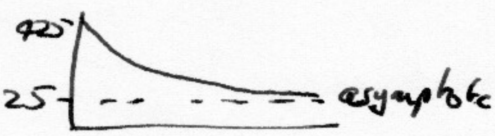
$= -20e^{-0.05t}$

At $t = 50$, $\frac{dT}{dt} = -20e^{-0.05\times 50} = -1.64\,°C/\text{min}$

so rate of decrease $= 1.64\,°C/\text{minute}$.

(d) $e^{x} > 0$ for all real x, so $T > 25$ for all x

$\Rightarrow$ can never get down to $20°C$



5. (a) At minimum, $dy/dx = 0$.

$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\sin x(\cos x)^{-1}\right) = \cos x(\cos x)^{-1} + \sin x(-1)(\cos x)^{-2} \times (-\sin x)$

$= 1 + \tan^2 x = \sec^2 x$.

$\frac{d}{dx}(2x-1)\tan 2x = 2\tan 2x + (2x-1)(2\sec^2 2x)$

$\therefore\ 2\tan 2k + (2k-1)(2\sec^2 2k) = 0$

$= \left(\frac{2}{\cos 2k}\right)\left(\sin 2k + \frac{2k-1}{\cos 2k}\right)$

5+

Think: $\sin 2A = 2\sin A\cos A$,

so $\sin 4k = 2\sin(2k)\cos(2k)$, so write $\tan 2k$

required

and $\sec^2 2k$ in terms of $\sin 2k$ & $\cos 2k$

$\therefore$ either $\left(\frac{2}{\cos 2k}\right) = 0$ (not possible)

or $\sin 2k + \frac{2k-1}{\cos 2k} = 0$.

$\times \cos 2k$:

$\sin 2k \cos 2k + 2k - 1 = 0$

$= \frac{1}{2}\sin 4k + 2k - 1$

$\times 2$:

$\therefore 4k + \sin 4k - 2 = 0$.

(b) $0.3 =$

$(2 - \sin(4 \times \text{ANS})) \div 4 =$ (calculator on radians)

$x_1 = 0.2670$ (round to 4 d.p.)

$x_2 = 0.2809$

$x_3 = 0.2746$

$x_4 = 0.2774$

(c) If it is 0.277 to 3 s.f., it must lie between 0.2765 and 0.2775.

Let $f(k) = 4k + \sin 4k - 2$

$f(0.2765) = -8.7 \times 10^{-5}$

$f(0.2775) = 5.7 \times 10^{-3}$

sign change and $f(x)$ is continuous, so the root lies in this interval

6. (a) $\sin^2\theta + \cos^2\theta \equiv 1$

$\div \sin^2\theta$:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}, \quad 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta,$$

$$\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$$

(b) Spot that $\operatorname{cosec}^4\theta - \cot^4\theta$ = "difference of two squares"

$$\equiv (\operatorname{cosec}^2\theta + \cot^2\theta)(\operatorname{cosec}^2\theta - \cot^2\theta)$$

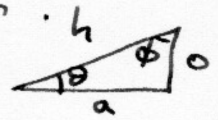
$$\equiv (\operatorname{cosec}^2\theta + \cot^2\theta) \quad \text{since} \quad \operatorname{cosec}^2\theta - \cot^2\theta = 1.$$

(c) From above, can rewrite as

$$\operatorname{cosec}^2\theta + \cot^2\theta = 2 - \cot\theta.$$

Think: this has two trig. functions, $\operatorname{cosec}\theta$ and $\cot\theta$. We must convert to a "one trig. function" equation.

We know $\sec^2\theta = 1 + \tan^2\theta$ so $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

[

$\tan\theta = \cot\phi$, $\operatorname{cosec}\theta = \sec\phi$

$\Rightarrow \operatorname{cosec}^2\phi = 1 + \cot^2\phi$, then let $\phi = \theta$].

$$1 + \cot^2\theta + \cot^2\theta = 2 - \cot\theta$$

(a quadratic, so write as $= 0$)

$$2\cot^2\theta + \cot\theta - 1 = 0$$

Factors of -2 adding to 1 are $2 \times (-1)$:

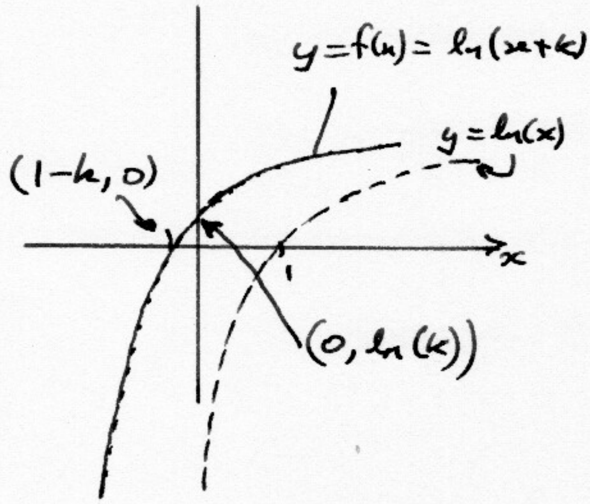
$$2\cot^2\theta + 2\cot\theta - \cot\theta - 1 = 2\cot\theta(\cot\theta + 1) - (\cot\theta + 1) = 0$$

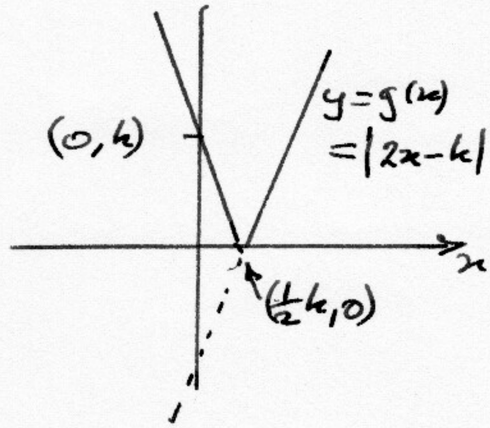
$$= (2\cot\theta - 1)(\cot\theta + 1), \quad \cot\theta = -1 \text{ or } +\tfrac{1}{2}.$$

For $90 < \theta < 180$, $\cot\theta$ is negative

$\Rightarrow \cot\theta = -1$, $\underline{\theta = 135°}$.

7. (a)





(b) $f(x) \in R$.

(c) $g\left(\frac{k}{4}\right) = \left|2\left(\frac{k}{4}\right) - k\right| = \left|\frac{k}{2} - k\right| = \left|\frac{-k}{2}\right| = \frac{k}{2}$ since $k > 1$.

$fg(k/4) = f\left(\frac{k}{2}\right) = \ln\left(k/2 + k\right) = \ln\left(\frac{3k}{2}\right) = \ln(k) + \ln(3/2)$

(d) Let $y = \ln(x+k)$.

$dy/dx = \frac{1}{x+k}$

Line $9y = 2x+1 \Rightarrow y = 2/9\, x + 1/9$,
gradient $2/9$.

$\therefore$ At $x = 3$, $dy/dx = \frac{1}{3+k} = 2/9$

$3 + k = 9/2$, $k = 4\tfrac{1}{2} - 3 = 1\tfrac{1}{2}$

8 (a) $\cos A = 3/4$.

$\sin^2 A + \cos^2 A = 1 \Rightarrow \sin A = \sqrt{1-\cos^2 A}$

$= \pm\sqrt{1-\frac{9}{16}} = \pm\sqrt{\frac{7}{16}}$

$= \frac{\pm\sqrt{7}}{4}$

S	A
T	C

4th quadrant, sin A negative

$\Rightarrow \sin A = \frac{-\sqrt{7}}{4}$

Then use $\sin 2A = 2\sin A\cos A$

$= 2\left(\frac{-\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = \frac{-3\sqrt{7}}{8}$

(b)(i) Using $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$,

and $\cos \pi/3 = 1/2$,

$\cos(2x+\pi/3) + \cos(2x-\pi/3)$

$\equiv \left(\cos 2x\left(\frac{1}{2}\right) - \sin 2x \sin \pi/3\right) + \left((\cos 2x)\,1/2 + \sin 2x \sin \pi/3\right)$

$\equiv \underline{\cos 2x}$.

(ii) $y = 3\sin^2 x + \cos 2x$ (using above identity)

$dy/dx = 3(2\sin x\cos x) + (-\sin(2x)) \times 2$

$= 3\sin 2x - 2\sin 2x$

$= \sin 2x$