JANUARY 2007 C3

1 (a) $\sin(A+B) = \sin A\cos B + \cos A \sin B$ *

$\therefore \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta \sin\theta$.

Also, $\sin 2\theta = 2\sin\theta\cos\theta$ *

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ * – from identities

$= (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$

$\therefore \sin 3\theta = (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= \underline{3\sin\theta - 4\sin^3\theta}$.

(b) With $\sin\theta = \frac{\sqrt{3}}{4}$,

$\sin 3\theta = \frac{3\sqrt{3}}{4} - 4\left(\frac{\sqrt{3}}{4}\right)^3 = \frac{3\sqrt{3}}{4} - \frac{3\sqrt{3}}{16}$

$= 3\sqrt{3}\left(\frac{4}{16} - \frac{1}{16}\right) = (3\sqrt{3}) \times \frac{3}{16} = \frac{9\sqrt{3}}{16}$

2.(a) $f(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}$

$= 1 + \frac{-3(x+2)+3}{(x+2)^2} = 1 + \frac{-3x-6+3}{(x+2)^2} = 1 + \frac{-3x-3}{(x+2)^2}$

$= \frac{x^2+4x+4-3x-3}{(x+2)^2} = \frac{x^2+x+1}{(x+2)^2}, \quad x \neq -2.$

2 (b) Discriminant $b^2 - 4ac = 1^2 - 4 = -3 < 0$ so
$x^2 + x + 1 = 0$ has no real roots.
$\therefore$ it never cuts the x-axis and, since >0 at
(eg) $x = 0$, it must be >0 everywhere.

or Let $y = x^2 + x + 1$, $dy/dx = 2x + 1$, $d^2y/dx^2 = 2$
Minimum at $2x + 1 = 0$, $x = -1/2$, $y = 1/4 - 1/2 + 1 = 3/4$
and this is a minimum because d^2y/dx^2 positive.

(c) $(x+2)^2$ is positive for all values of x $(x \neq 2)$
$\therefore$ $\dfrac{x^2 + x + 1}{(x+2)^2}$ must be >0.

3. $x = 2\sin y$

(a) $\sin \pi/4 = \dfrac{1}{\sqrt{2}}$, $2 \sin \pi/4 = \sqrt{2}$
so $(\sqrt{2}, \pi/4)$ is on C.

(b) Though we could write $y = \sin^{-1}(x/2)$ you don't know how to differentiate this.
Instead, use $\dfrac{dx}{dy} = 2\cos y$

$$\frac{dy}{dx} = \frac{1}{2\cos y}.$$

Then at P, $dy/dx = \dfrac{1}{2\cos \pi/4} = \dfrac{1}{2\left(\frac{1}{\sqrt{2}}\right)} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

(c) Gradient of normal $= \dfrac{-1}{(1/\sqrt{2})} = -\sqrt{2}$

$y - \pi/4 = -\sqrt{2}(x - \sqrt{2}) = -\sqrt{2}x + 2$

$y = -\sqrt{2}x + (2 + \pi/4)$

4.(1) Curve C is $y = \frac{\cancel{9}x}{9+x^2} = \frac{u}{v}$, $u = x$, $V = 9+x^2$

$$\frac{dy}{dx} = \frac{u'V - uV'}{V^2} = \frac{1(9+x^2) - x(2x)}{(9+x^2)^2} = \frac{9-x^2}{(9+x^2)^2}$$

At $\frac{dy}{dx} = 0$, $9 - x^2 = 0$, $x^2 = 9$, $x = \pm 3$

$\therefore$ Turning points are $\left(+3, \frac{3}{9+9}\right) = \left(3, \frac{1}{6}\right)$

and $\left(-3, \frac{-3}{9+9}\right) = \left(-3, -\frac{1}{6}\right)$

(ii) $y = \left(1+e^{2x}\right)^{3/2}$

$$dy/dx = 3/2\left(1+e^{2x}\right)^{1/2}\left(2e^{2x}\right) = 3e^{2x}\left(1+e^{2x}\right)^{1/2}$$

At $x = \frac{1}{2}\ln 3$, $e^{2x} = e^{\ln 3} = 3$

$$dy/dx = 3 \times 3(1+3)^{1/2} = 9\sqrt{4} = 18$$

5.

(a) $y = \sqrt{3}\cos x + \sin x$

Let $y = R\sin(x+\alpha) = R\cos\alpha \sin x + R\sin\alpha\cos x$

$= 1\sin x + \sqrt{3}\cos x$

$\Rightarrow R\sin\alpha = \sqrt{3}$, $R\cos\alpha = 1$

$R^2 = (\sqrt{3})^2 + 1^2 = 3 + 1 = 4$, $R = 2$.

$\cos\alpha = 1/R = \frac{1}{2}$, $\alpha = \pi/3$

(or $\tan\alpha = \frac{\sqrt{3}}{1}$, $\alpha = \pi/3$)

$\therefore y = 2\sin(x + \pi/3)$

(b) At $y = 1$, $\sin(x+\pi/3) = y/2 = 1/2$, $(x + \pi/3) = \pi/6 + n2\pi$

or $5\pi/6 + n2\pi$

$$\begin{array}{c|c} S & A \\ \hline T & C \end{array}$$

$\therefore x = \frac{\pi}{6} - \pi/3 + 2\pi = \frac{11}{6}\pi$

or $x = 5\pi/6 - \pi/3 = \frac{3\pi}{6} = \frac{1}{2}\pi$

6.(a) $f(x) = \ln(4-2x)$, $x<2$

Let $y = \ln(4-2x)$.

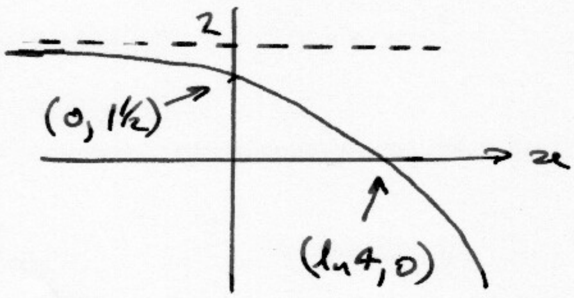
swap x & y, $x = \ln(4-2y)$, make y the subject

$e^x = 4-2y$, $2y = 4-e^x$,

$y = 2 - \frac{1}{2}e^x = f^{-1}(x)$. Domain $x \in R$.

(b) Range of f^{-1} = domain of f, $\underline{x \in R,\ x<2}$.

(c)



At $y=0$, $e^x=4$.

(d) $x_{n+1} = -\frac{1}{2}e^{x_n}$, $x_0 = -0.3$, $x_1 = -0.3704$,

$x_2 = -0.3452$, $x_3 = -0.3540$

(e) Many iterations, $x \to -0.3517 \Rightarrow k$ (exact solution)

To prove $k = -0.352$ to 3dp:

where $y = x+2$ and $y = 2-\frac{1}{2}e^x$ cross,

$(x+2) - (2-\frac{1}{2}e^x) = x + \frac{1}{2}e^x = 0$

$=$ "$g(x)$".

$$\left.\begin{array}{l} g(-0.3515) = 3.16\times10^{-4} \\ g(-0.3525) = -1.04\times10^{-3} \end{array}\right\}$$ sign change and $g(x)$ continuous so $g(x)=0$ has a root in this interval.

7. $f(x) = x^4 - 4x - 8$

(a) $f(-1) = (-1)^4 - 4(-1) - 8 = 1 + 4 - 8 = -3$

$f(-2) = (-2)^4 - 4(-2) - 8 = 16 + 8 - 8 = 16$

$\Big\}$ Sign change & $f(x)$ is continuous $\Rightarrow f(x) = 0$ between $x = -1$ and -2.

(b) Turning point is where $dy/dx = 0$.

$dy/dx = 4x^3 - 4$.

At $dy/dx = 0$, $4x^3 - 4 = 0$, $x^3 = 4/4 = 1$, $x = 1$.

Then $y = f(1) = 1^4 - 4 - 8 = -11$

Turning point is at $(1, -11)$.

(c)

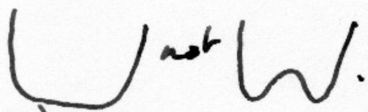
$$
\begin{array}{rl}
 & x^3 + 2x^2 + 4x + 4 \\
(x-2) & \overline{)\, x^4 + 0x^3 + 0x^2 - 4x - 8} \\
 & \underline{x^4 - 2x^3} \quad - \\
 & +2x^3 + 0x^2 - 4x - 8 \\
 & \underline{2x^3 - 4x^2} \quad - \\
 & 4x^2 - 4x - 8 \\
 & \underline{4x^2 - 8x} \quad - \\
 & 4x - 8 \\
 & \underline{4x - 8} \quad - \\
 & 0
\end{array}
$$

$= x^3 + ax^2 + bx + c$,

$a = 2$,

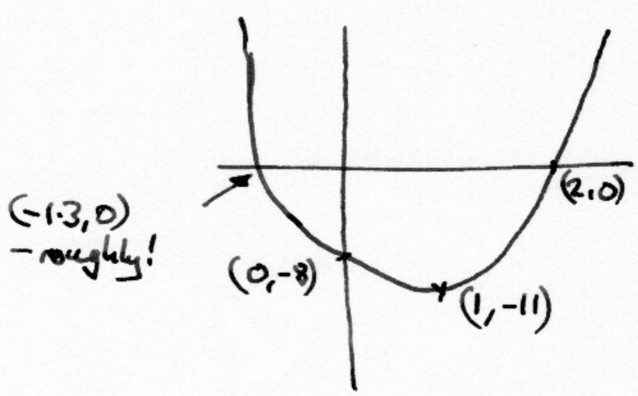
$b = 4$,

$c = 4$

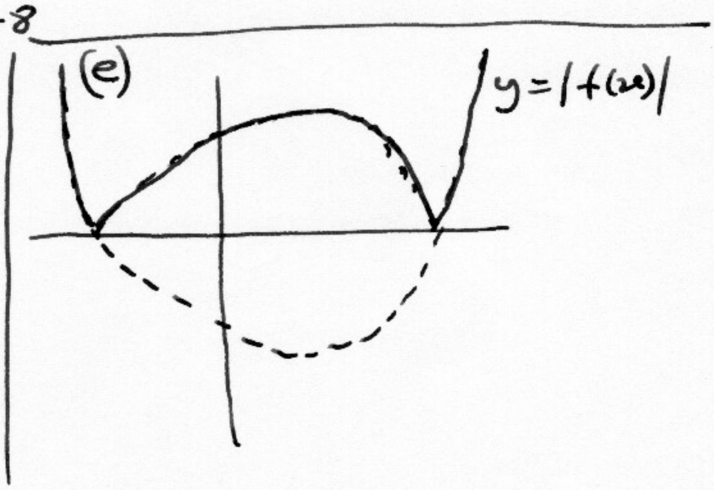
(d) Let $y = f(x)$. Only one turning point so  (both 4th order curves).

~~Think: $dy/dx = 4x^3 - 4$~~

At $x = 0$, $dy/dx = -4$ and $y = -8$



(e)



8 (i) Think of identities that link $\sec^2 x$ & $\tan^2 x$:

$\sec^2 x = 1 + \tan^2 x$

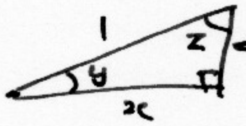
$\operatorname{cosec}^2 x = 1 + \cot^2 x$ (as above if one puts $y = 90 - x$, $\sec x = \operatorname{cosec} y$, $\tan x = \cot y$, then write in terms of x rather than y)

Then $\sec^2 x - \operatorname{cosec}^2 x \equiv 1 + \tan^2 x - (1 + \cot^2 x)$

$$\equiv \tan^2 x - \cot^2 x$$

(ii) $y = \arccos x \quad (= \cos^{-1}(x))$

(a) $\cos y = x$



$z = \pi/2 - y$

$\cancel{\sin y = \sqrt{1 - x^2}}$

$\sin z = x, \quad z = \arcsin(x)$

$$= \underline{\underline{\pi/2 - y}}.$$

(b) $\arccos x + \arcsin x = y + (\pi/2 - y) = \underline{\underline{\pi/2}}.$