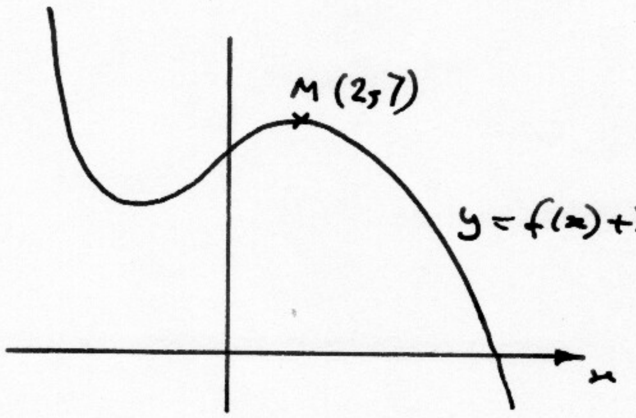
C3 January 2006

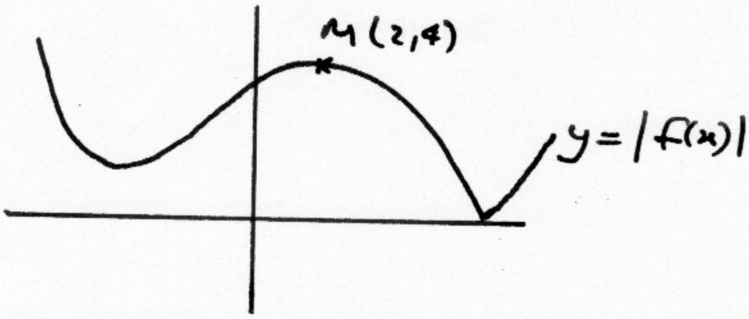
1.

(a)



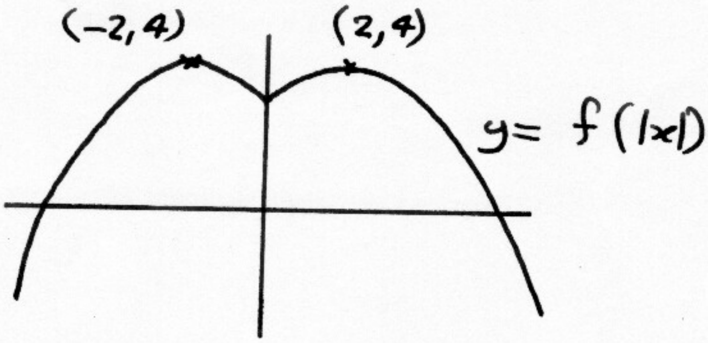
y increases by 3
- just moves up

(b)



Think: negative y values become positive

c)



Think: y the same for negative x as for positive x

2.

$$\frac{2x^2+3x}{(2x+3)(x-2)} - \frac{6}{x^2-x-2} = \frac{x\cancel{(2x+3)}}{(x-2)\cancel{(2x+3)}} - \frac{6}{(x+1)(x-2)}$$

$$= \frac{x(x+1)-6}{(x+1)(x-2)} = \frac{x^2+x-6}{(x+1)(x-2)} = \frac{(x+3)(x-2)}{(x+1)(x-2)} = \frac{x+3}{x+1}$$

3. $y = \ln(\tfrac{1}{3}x)$

Think: need to know:
* coordinates of a point on the normal
* gradient of tangent there
* hence gradient of normal

At $x = 3$, $y = \ln(\tfrac{3}{3}) = \ln(1) = 0$

$$dy/dx = \left(\frac{1}{(\tfrac{1}{3}x)}\right) \times \left(\frac{1}{3}\right) = \frac{1}{x}$$

$\Rightarrow$ at $x = 3$, $dy/dx = \tfrac{1}{3}$ = gradient of curve

$\Rightarrow$ gradient of normal is $\dfrac{-1}{(\tfrac{1}{3})} = -3$

Use $y - y_1 = m(x - x_1)$ (equation for straight line)

$$y - 0 = -3(x-3) = -3x + 9$$

$\therefore y = -3x + 9$

4(a)(i) $$\frac{d}{dx}\left(x^2 e^{3x+2}\right) = 2x e^{3x+2} + x^2 e^{3x+2}(3)$$
$$= (3x^2 + 2x)e^{3x+2}$$

Think: $x^2 e^{3x+2} = uv$, $du/dx = 2x$,
$\dfrac{dv}{dx} = 3e^{3x+2}$, $d/dx(uv) = u'v + uv'$
(product rule)

4(a) (ii) Either use quotient rule or product rule.

Quotient: $u(x) = \cos(2x^3)$, $du/dx = -\sin(2x^3)(6x^2)$

think: chain rule,
$d/dt(\cos t) = -\sin t$,
$d/dx(2x^3) = 6x^2$

$v(x) = 3x$, $dv/dx = 3$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(3x)(6x^2)(-\sin(2x^3)) - \cos(2x^3)(3)}{(3x)^2}$$

$$= \frac{-18x^3\sin(2x^3) - 3\cos(2x^3)}{9x^2}$$

$$= -2x\sin(2x^3) - \frac{\cos(2x^3)}{3x^2}$$

or product

Let $u = \cos(2x^3)$ as above,

$v = 1/3\, x^{-1}$, $\frac{dv}{dx} = -\frac{1}{3}x^{-2}$

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$= -6x^2\sin(2x^3)(1/3\, x^{-1}) + \cos(2x^3)(-1/3\, x^{-2})$$

$$= -2x\sin(2x^3) - \frac{\cos(2x^3)}{3x^2}$$

4 (b) Think: although I could write

$$2y+6 = \sin^{-1}(x/4), \quad y = 1/2\left(\sin^{-1}(x/4) - 6\right)$$

you don't know how to differentiate this (it is an FM topic!)

Instead differentiate with respect to y (giving dx/dy), then take reciprocal.

$$x = 4\sin(2y+6)$$

$$\frac{dx}{dy} = 4\cos(2y+6)(2) = 8\cos(2y+6)$$

$$\frac{dy}{dx} = \frac{1}{8\cos(2y+6)}$$

Question asks for it <u>in terms of x</u> so either:

(a) Put $x/4 = \sin(2y+6)$,

$$2y+6 = \sin^{-1}(x/4)$$

$$\frac{dy}{dx} = \frac{1}{8\cos\left(\sin^{-1}(x/4)\right)}$$

or (b) $\sin^2(2y+6) + \cos^2(2y+6) = 1$ (identity),

$$\cos(2y+6) = \sqrt{1-\sin^2(2y+6)} = \sqrt{1-(x/4)^2}$$

$$\frac{dy}{dx} = \frac{1}{8\sqrt{1-(x/4)^2}}$$

[PS. dy/dx may be positive or negative depending on y – a formula in terms of x will not tell you which! You don't need to know this].

5. (a) $f(x) = 2x^3 - x - 4$

$f(x) = 0 \Rightarrow 2x^3 - x - 4 = 0$

We need to get an expression for x^2, then square root it.

$2x^3 = x + 4$,

$\div 2x$ $\quad x^2 = \frac{x}{2x} + \frac{4}{2x} = \frac{1}{2} + \frac{2}{x}$

$= \frac{2}{x} + \frac{1}{2}$

$\therefore x = \cancel{\sqrt{\frac{2}{4} + \frac{1}{2}}}$

$\sqrt{\frac{2}{x} + \frac{1}{2}}$

(b) Calculator, $1.35 =$

~~$\sqrt{(ANS}$~~

$\sqrt{(2 \div ANS + 1 \div 2)} =$

$x_1 = 1.4077$, (1.41 to 2 d.p.)

$x_2 = 1.3859$ (1.39)

$x_3 = 1.3939$ (1.39)

(c) If $\alpha = 1.392$ to 3 d.p. it must lie between 1.3915 and 1.3925.

$$\left.\begin{aligned} f(1.3915) &= -2.85 \times 10^{-3} \\ f(1.3925) &= 7.77 \times 10^{-3} \end{aligned}\right\}$$

sign change and $f(x)$ is continuous, so the root lies between these values and will round to 1.392.

Calculator: do $1.3915 =$

$2 \times ANS \boxed{x^3} - ANS - 4 =$

then $1.3925 =$

and recall previous $2 \times ANS \ldots$ line

6. (a) $f(x) = 12\cos x - 4\sin x$

Let $f(x) = R\cos(x+\alpha)$, $0 \leq \alpha \leq 90$, $R > 0$.

$$= R\cos\alpha\cos x - R\sin\alpha\sin x$$
(identity).

$\therefore$ We require $R\cos\alpha = 12$, $R\sin\alpha = +4$
$(-R\sin\alpha = -4)$

$$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{12} = \tfrac{1}{3},$$

$\alpha = \tan^{-1}(\tfrac{1}{3}) = 18.43°$ (to 2d.p.) (check calculator on <u>degrees</u>).

$(R\sin\alpha)^2 + (R\cos\alpha)^2 = R^2 = 12^2 + 4^2 = 144 + 16 = 160,$

$R = \sqrt{160} = 12.65$ (to 2 d.p.).

(b) $12\cos x - 4\sin x = \sqrt{160}\cos(x+\alpha) = 7$

$\cos(x+\alpha) = \frac{7}{\sqrt{160}},$

$x + \alpha = \cos^{-1}\left(\frac{7}{\sqrt{160}}\right) = 56.4°$ or -56.4 $(+ n360)$.

[nb. calculate like this, do not round $\frac{7}{\sqrt{160}}$ to decimals first!]

S	A ✓
T	C ✓ ← cos positive

$\therefore x = 56.4 - 18.43 = 37.96 + n360$

or $-56.4 - 18.43 = -74.83 + n360$

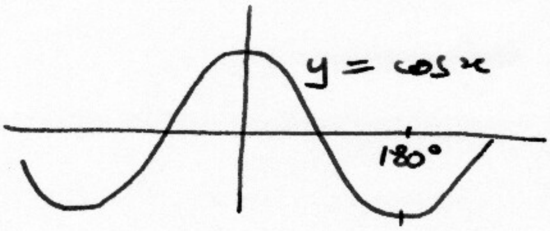
for $0 \leq x < 360$, $x = 37.96°$ or $360 - 74.83$
$= 285.17°$

$=$ <u>38.0</u> or <u>285.2°</u> to 1d.p.

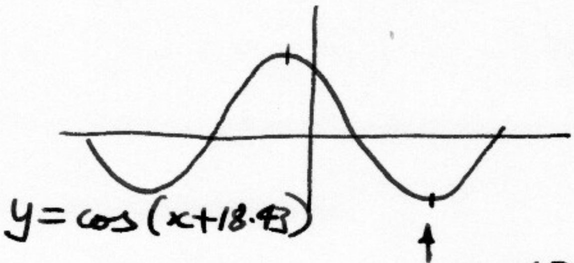
6(c)(i) $12\cos x - 4\sin x = \sqrt{160}\cos(x + 18.43°)$,

minimum value $-\sqrt{160} = -12.65$

(ii)



shifted ← 18.43°



$x = 180 - 18.43$

$= \underline{161.57°}$

7.
(a)(i)

Think: we want to start with the left hand side (LHS) and turn it into the RHS.
The RHS is a function of x, the LHS has a function of $2x$ so we must start by converting $\cos 2x$ into a function of x.

$\cos(x+x) = \cos x \cos x - \sin x \sin x$

$\cos 2x = \cos^2 x - \sin^2 x$

Now note that we have $\cos x + \sin x$, $\cos x - \sin x$ expressions so factorise $\cos^2 x - \sin^2 x$ (difference of two squares) $= (\cos x + \sin x)(\cos x - \sin x)$

Then LHS $= \dfrac{\cos 2x}{\cos x + \sin x} = \dfrac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)}$

$= \cos x - \sin x = \text{RHS}$ ✓.

7 (a) (ii) $\sin 2x = \sin(x+x) = \sin x \cos x + \cos x \sin x$

$= 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

Seeing that we don't have $\sin^2 x$ on the RHS,

put (from $\sin^2 x + \cos^2 x = 1$) $\sin^2 x = 1 - \cos^2 x$,

$\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$

Then $\cos 2x - \sin 2x = 2\cos^2 x - 2 \sin x \cos x - 1$

$\frac{1}{2}(\cos 2x - \sin 2x) = \cos^2 x - \sin x \cos x - \frac{1}{2}$

$= RHS$.

(b) "Hence" means using the identities given in (a).
[you can use these even if you did not manage to prove them].

$\cos\theta \left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \cos\theta(\cos\theta - \sin\theta)$ (from a(i))

$= \cos^2\theta - \cos\theta \sin\theta$.

Put

$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) - \frac{1}{2} = 0$ (original equation, as $= 0$)

$= \cos^2\theta - \cos\theta \sin\theta - \frac{1}{2}$

$= \frac{1}{2}(\cos 2\theta - \sin 2\theta)$ (from (a)(ii)).

$\therefore \cos 2\theta - \sin 2\theta = 0$, $\sin 2\theta = \cos 2\theta$,

$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = 1$

(c) $2\theta = \tan^{-1}(1) = \pi/4$ (45°) or $\pi + \pi/4 = 5\pi/4$, $\begin{array}{c|c} S & A\checkmark \\ \hline \checkmark T & C \end{array}$

i.e. $2\theta = \pi/4 + n\pi$, $\theta = \pi/8 + n\pi/2 = \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$.

8. $f(x) = 2x + \ln 2$

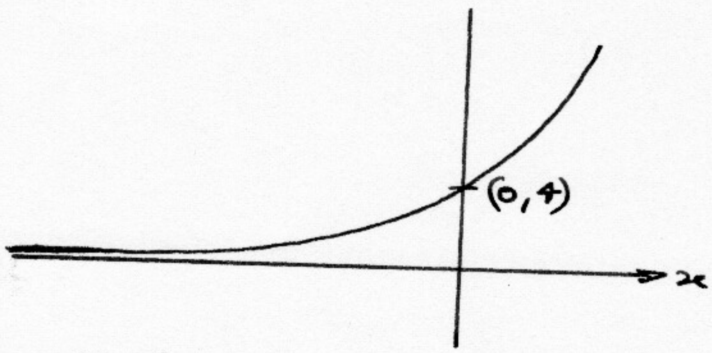
$g(x) = e^{2x}$

(a) $g(f(x)) = e^{2(2x+\ln 2)}$

$= e^{4x} e^{2\ln 2} = e^{4x}\left(e^{\ln 2}\right)^2$

$= 2^2 e^{4x} = 4e^{4x}$

(b)



At $x = 0$,
$y = 4e^0 = 4$.

(c) $g(f(x)) > 0$

(d) $\frac{d}{dx}\left[gf(x)\right] = \frac{d}{dx}\left(4e^{4x}\right) = 16e^{4x}$

If $16e^{4x} = 3$, $e^{4x} = 3/16$,

$4x = \ln\left(\frac{3}{16}\right)$, $x = \frac{1}{4}\ln\left(\frac{3}{16}\right)$

$= -0.41849$

$= -0.418$ to 3d.p.