C2 January 2007

1. (a) $f(x) = x^3 + 3x^2 + 5$, find $f''(x)$.

$$f'(x) = \frac{df}{dx} = 3x^2 + 3(2x) = 3x^2 + 6x$$

$$f''(x) = \frac{d^2f}{dx^2} = 6x + 6$$

(b) $$\int_1^2 f(x)\,dx = \int_1^2 x^3 + 3x^2 + 5\,dx$$

$$= \left[\frac{x^4}{4} + 3\left(\frac{x^3}{3}\right) + 5x\right]_1^2 = \left[\frac{x^4}{4} + x^3 + 5x\right]_1^2$$

$$= \left(\frac{16}{4} + 8 + 10\right) - \left(\frac{1}{4} + 1 + 5\right)$$

$$= 22 - 6\tfrac{1}{4} = 15\tfrac{3}{4}$$

2. (a) $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + \dots$

(formula book)

$a = 1$ so a^n, a^{n-1} etc all $= 1$

$b = -2x$, $n = 5$

$$\therefore (1-2x) = 1 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + \dots$$

$$= 1 + 5(-2x) + 10(4x^2) + 10(-8x^3) + \dots$$ shows terms have been omitted

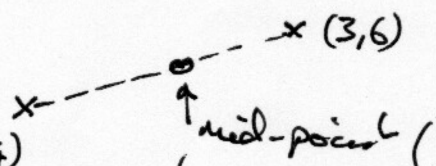
$$= 1 - 10x + 40x^2 - 80x^3 + \dots$$

(b) $$(1+x)(1-2x)^5 \approx (1+x)(1-10x)$$

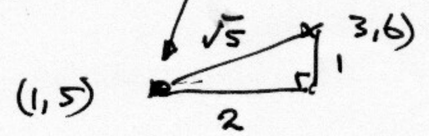
$$= 1 - 10x + x - 10x^2$$

$$= 1 - 9x$$, ignoring x^2 and higher terms.

3.



mid-point $\left(\frac{-1+3}{2}, \frac{4+6}{2}\right) = (1,5)$



Centre $(1,5)$, radius $\sqrt{5}$,

$\therefore$ Circle C is

$$(x-1)^2 + (y-5)^2 = (\sqrt{5})^2 = 5.$$

(from $(x-a)^2 + (y-b)^2 = r^2$ formula).

4. $5^x = 17$

Either: take logs of each side,

$$\log(5^x) = x \log 5 = \log 17 \quad \text{(any base)}$$

$$\therefore x = \frac{\log 17}{\log 5} = 1.7604$$

$$= \underline{1.76} \text{ to 3 sig. figs}$$

or $x = \log_5(17) = 1.76$ (calculator).

5. (a) Factor theorem: if $(x-a)$ is a factor of $f(x)$, then $f(a) = 0$ (& vice-versa).

Let $x+2 = x-a \Rightarrow a = -2$. $f(x) = x^3 + 4x^2 + x - 6$.

$$f(a) = f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$$

$$= -8 + 16 - 2 - 6 = 0$$

n.b. Do not do $-2 \boxed{x^2}$ on your calculator! Do it in your head, do $2 \boxed{x^2}$ or use the ANS button.

(b) Now use long division to convert the cubic to a quadratic, then factorise it.

$$\begin{array}{r} x^2 + 2x - 3 \\ x+2 \overline{\smash{)}\, x^3 + 4x^2 + x - 6} \\ \underline{x^3 + 2x^2} \quad - \\ 2x^2 + x - 6 \\ \underline{2x^2 + 4x} \quad - \\ -3x - 6 \\ \underline{-3x - 6} \\ 0 \end{array}$$

$$x^2 + 2x - 3 = (x+3)(x-1)$$

$$\therefore f(x) = (x+2)(x+3)(x-1) \quad \text{"final answer"}$$

(c) $f(x) = 0 \quad \therefore x+2 = 0 \Rightarrow x = -2$

or

$x+3 = 0 \Rightarrow x = -3$

or

$x-1 = 0 \Rightarrow x = 1$.

6. $2\cos^2 x + 1 = 5\sin x \quad , \quad 0 \le x < 2\pi$

↑ put calculator in radians mode.

"Get in terms of 1 type of trig. function".

→ we cannot convert sin to cos, but we can convert $\underline{\cos^2 x \text{ to } \sin^2 x}$.

$\cos^2 x + \sin^2 x \equiv 1$ (identity)

$\therefore \cos^2 x = 1 - \sin^2 x$

$2(1 - \sin^2 x) + 1 = 5\sin x$

$2\sin^2 x + 5\sin x - 3 = 0$

Let $y = \sin x$, $\quad 2y^2 + 5y - 3 = 0$ quadratic.

$ac = -6$, factors 6×-1

$2y^2 + 6y - y - 3 = 0$

$= 2y(y+3) - 1(y+3)$

$= (2y-1)(y+3)$

or just

$(2y \quad -\tfrac{1}{1})(1y \quad + \tfrac{6}{2})$

$= (2y-1)(y+3)$

$\therefore \sin x + 3 = 0$, $\sin x = -3$ ("no real roots")

or

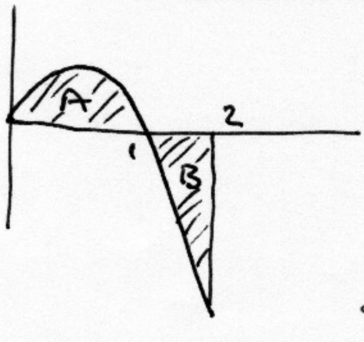
$2\sin x - 1 = 0$, $\sin x = \tfrac{1}{2}$,

$x = \pi/6$ or $\pi - \pi/6$

$= \pi/6, \ 5\pi/6$

S ✓	A ✓
T	C

7.



For "total area", we must find areas A & B, make both positive, then add.

$y = x(x-1)(x-5) = x(x^2 - 6x + 5)$

$= x^3 - 6x^2 + 5x$

Area A $= \int_0^1 x^3 - 6x^2 + 5x \, dx = \left[\frac{x^4}{4} - 2x^3 + \frac{5x^2}{2}\right]_0^1 = \left(\frac{1}{4} - 2 + \frac{5}{2}\right) - 0$

$= 2\tfrac{3}{4} - 2 = \tfrac{3}{4}$.

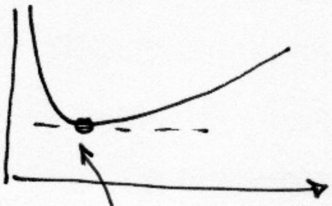
For Area B $= \int_1^2 x^3 - 6x^2 + 5x \, dx = \left[\frac{x^4}{4} - 2x^3 + \frac{5}{2}x^2\right]_1^2 = \left(\frac{16}{4} - 16 + 10\right) - \frac{3}{4} = -2\tfrac{3}{4}$

→ make positive, area B $= +2\tfrac{3}{4}$.

$\therefore$ $\underline{\text{Total} = \tfrac{3}{4} + 2\tfrac{3}{4} = 3\tfrac{1}{2}}$

8. $C = \frac{1400}{v} + \frac{2v}{7} = 1400v^{-1} + \frac{2}{7}v$

(a)



Minimum when $\frac{dc}{dv} = 0$

$\frac{dc}{dv} = -1400v^{-2} + \frac{2}{7}$

At $dc/dv = 0$,

$1400v^{-2} = 2/7$

$\times \frac{7}{2}v^2$ each side:

$4900 = v^2$, $\Rightarrow v = 70$ km/hr

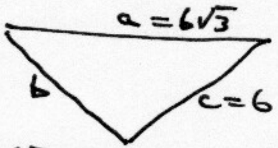
(b) $\frac{d^2C}{dv^2} = \frac{d}{dv}(-1400v^{-2} + 2/7) = 2800v^{-3}$.

At $v = 70$, $\frac{d^2C}{dv^2}$ is positive $\therefore$ C is a minimum here.

(c) Substitute $v = 70$ into $C = \frac{1400}{v} + \frac{2v}{7}$

$C = \frac{1400}{70} + \frac{2 \times 70}{7} = 20 + 20 = \underline{£40}$.

9. (a) Cosine rule



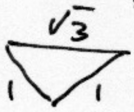
$a^2 = b^2 + c^2 - 2bc\cos A$ (formula book)

$\therefore \frac{b^2 + c^2 - a^2}{2bc} = \cos A$

$\cos A = \frac{6^2 + 6^2 - (6\sqrt{3})^2}{2 \times 6 \times 6} = \frac{36 + 36 - 108}{72}$

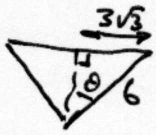
$= \frac{-36}{72} = -1/2$

[or $\div 6$



, $\cos A = \frac{1 + 1 - (\sqrt{3})^2}{2} = -\frac{1}{2}$]

[or divide



, $\sin\theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \therefore \theta = \frac{\pi}{3}$]

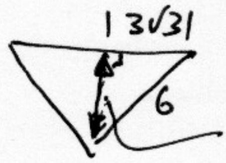
Then $A = \cos^{-1}(-1/2) = \frac{2\pi}{3}$ = angle POR.

(b) Area of a sector $= \frac{1}{2}r^2\theta = 1/2 \times 6^2 \times \frac{2\pi}{3}$

$= \frac{36\pi}{3} = 12\pi$ m^2.

(c) Area of triangle $= \frac{1}{2}bc\sin A = \frac{1}{2} \times 6^2\left(\frac{\sqrt{3}}{2}\right) = 9\sqrt{3}$ m^2

[or



$\sqrt{6^2 - (3\sqrt{3})^2} = \sqrt{36 - 27} = \sqrt{9} = 3$

area $= 1/2\, wh = 1/2(6\sqrt{3})(3) = 9\sqrt{3}$ m^2].

(d) Area of segment = sector − triangle $= 12\pi - 9\sqrt{3}$

$= 22.1$ m^2

(e) Arc length $= r\theta$, perimeter $6 + 6 + 6 \times \frac{2\pi}{3} = 12 + 4\pi = \underline{24.6\text{ m}}$

10.

(a) Define $S_n = a + ar + ar^2 + \dots ar^{n-1}$ (n terms)

$rS_n = \quad ar + ar^2 + \dots ar^{n-1} + ar^n$.

subtracting:

$rS_n - S_n = ar^n - a = a(r^n - 1)$

$= (r-1)S_n$

$\therefore S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{a(1-r^n)}{1-r}$.

(b)

$\sum_{k=1}^{10} 100(2^k)$ means $100 \times 2^1 + 100 \times 2^2 + \dots 100 \times 2^{10}$

First term $a = 200$

Common ratio $r = 2$

10 terms.

$= S_{10} = \dfrac{200(1-2^{10})}{1-2} = 204600$

(c)

Another series, $\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \dots$

$= \frac{5}{6} + \frac{5}{6} \times \frac{1}{3} + \frac{5}{6} \times \left(\frac{1}{3}\right)^2 + \dots$

$a = 5/6$, $r = \frac{1}{3}$

$S_\infty = \dfrac{a}{1-r} = \dfrac{5/6}{1-1/3} = \dfrac{(5/6)}{(2/3)} = 3/2 \times \frac{5}{6}$

$= \frac{15}{12} = 1\frac{1}{4}$.